EXACT TESTING PROCEDURES FOR UNBALANCED RANDOM AND MIXED LINEAR MODELS

By

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To my mother and the memory of my father

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Although much attention has been given to the problem of estimating variance components in unbalanced random and mixed linear models, the same cannot be said about the development of exact testing procedures for the aforementioned quantities. One reason for this is the mathematical complexity of the problem. Even in some balanced random or mixed models, no exact analysis of variance tests exist for testing certain variance components. Consequently, approximate methods, such as Satterthwaite's procedure, are typically relied upon to furnish the required tests. Unfortunately, the exact distributions of these approximate procedures are generally unknown, even under the null hypothesis.

Recently, some progress has been made in the construction of exact tests for the variance components and for estimable linear functions of the fixed-effects parameters in unbalanced random and mixed models. In this dissertation we extend some of these known results and generate some new ones. Specifically, exact procedures are developed to test hypotheses concerning the variance components in an unbalanced random two-way model with some cells missing, an unbalanced random 3-fold nested model, and an unbalanced random s-fold nested model where the imbalance affects the last two stages only. Also, methods are derived to test hypotheses about the variance components and about estimable linear functions of the fixed-effects parameters in a general unbalanced mixed model with imbalance affecting the last stage only.

CHAPTER ONE
INTRODUCTION

Although much attention has been given to the problem of estimating variance components (see, e.g., Rao and Kleffe (1988)), the same cannot be said about the development of exact testing procedures concerning the variance components in unbalanced random and mixed linear models. Perhaps one reason for this is due to the mathematical complexity of the problem. Even for some balanced models no exact analysis of variance (ANOVA) tests exist for testing certain variance components. For example, in a three-way cross-classification random-effects model, no exact procedures exist for testing hypotheses concerning the variance components associated with the main effects of the model. Also, when exact tests do exist for balanced models, they generally cannot be used to test the corresponding hypotheses in unbalanced models. The reason for this is that for unbalanced models, the traditional test statistics used with balanced data are, in general, no longer based upon independent, chi-squared-type sums of squares.

Approximate methods, such as Satterthwaite's (1946) procedure, are typically relied upon to furnish tests for the variance components when no such exact procedure exists. However, approximate methods are just that -- approximate. Some may be relatively easy to implement, and perform reasonably well in certain situations, but they cannot be expected to always produce adequate solutions (see, e.g., Cummings and Gaylor (1974), Tietjen (1974), and more recently, Ames and Webster (1991)). Thus there is a clear need for the development of exact testing procedures.

In this dissertation we have concentrated on developing exact methodology for some unbalanced random and mixed linear models. Much of this work has been inspired by the derivations given in Khuri (1984, 1987, 1990), Khuri and Littell (1987), Gallo (1987), and Gallo and Khuri (1990). Their results and those obtained in this dissertation are based on certain transformations which reduce the analysis of the unbalanced model to that of a balanced one. A crucial theme in the construction of these transformations is a scheme for resampling from the error vector. In this scheme, only a portion of the error vector is utilized in making the transformation. Consequently, the values of the resulting test statistics will not be unique. However, the distributional properties of these statistics will remain unchanged no matter what portion of the error vector is used in the transformation. It is in this sense that our methodology is similar to procedures involving resampling. Appendix C provides an illustration of the utility of resampling as a means of “counteracting” the imbalance in a design.

This thesis is divided into four chapters. Chapters one and two give a brief introduction to the area of exact testing procedures in unbalanced random and mixed linear models and a review of the pertinent literature. The main contributions of this study are found in Chapter three. This chapter is divided into four sections. In the first section we develop exact tests for the variance components in a random two-way model with some cells missing. Section two deals with the unbalanced random 3-fold nested model. Here the imbalance is assumed to affect all stages of the design. In the last two sections we concentrate on more general models. The general mixed model with imbalance affecting the last stage only is analyzed in Section three. While in Section four we obtain exact procedures for testing the variance components in a random s-fold nested model with imbalance affecting the last two stages only. Finally, Chapter four summarizes our findings and offers directions for future research.

Throughout this work we will try to motivate, as much as possible, the reason behind each transformation we use. At times, the mathematical derivations may become "intense" in the sense that there may be pages containing nothing but equations. Unfortunately, there seems to be no way to avoid this. A numerical example is given at the end of each section in Chapter three to illustrate the proposed methodology. These examples also serve as useful (and probably needed) "rest stops" between pages full of formulas.

The following notation will be used throughout this thesis:

- rank(A) will denote the rank of the matrix A
- range(A) will denote the range space of the matrix A, that is, the vector space spanned by the columns of A.
- tr(A) will denote the trace of the matrix A.
- A^- will denote a generalized inverse of A, that is, any matrix which satisfies $AA^-A = A$.

- diag(A, B) will denote the matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. This can also be represented by $A \oplus B$ where "$\oplus$" is the direct sum operator.
- $A \otimes B$ will be used to represent the (right) direct (Kronecker) product of two matrices. Briefly, if A is $n \times m$ and B is $p \times q$ then $A \otimes B$ is the $np \times mq$ partitioned matrix whose $(i,j)^{th}$ matrix is $a_{ij}B$ where a_{ij} is the $(i,j)^{th}$ element of A.

CHAPTER TWO
LITERATURE REVIEW

The first paper that addresses the issue of exact tests for variance components in unbalanced models is due to Wald (1940). Wald constructed an exact test for $\sigma_\alpha^2/\sigma_\epsilon^2$ in the random one-way model:

$$y_{ij} = \mu + \alpha_i + \epsilon_{ij} \tag{2.1}$$

$(i = 1,2,\cdots,p;\ j = 1,2,\cdots,m_i)$, where μ is an unknown constant parameter; α_i and ϵ_{ij} are independent, normally distributed random variables with zero means and variances σ_α^2 and σ_ϵ^2, respectively. He later extended this result to the general random-effects model without interactions. Wald (1947) also considered the mixed model

$$\underset{\sim}{y} = X\underset{\sim}{\alpha} + Z\underset{\sim}{b} + \underset{\sim}{\epsilon} \tag{2.2}$$

where $\underset{\sim}{\alpha}$ is a vector of fixed effects; $\underset{\sim}{b} \sim N(\underset{\sim}{0},\ \sigma_b^2 I)$ independently of $\underset{\sim}{\epsilon} \sim N(\underset{\sim}{0},\ \sigma_\epsilon^2 I)$. Using a regression analysis approach, Wald was able to construct an exact test for $\sigma_b^2/\sigma_\epsilon^2$.

As pointed out in Spjøtvoll (1968), Wald's idea cannot be used to test variance components in all linear models. To illustrate this phenomenon, Spjøtvoll considered the random two-way model. Based on a modification of Wald's method he was able to derive an exact test for the variance component associated with the interaction effect. However, only under the assumption of additivity could he construct exact tests for the main effects variance components.

By modifying an approach suggested by Graybill and Hultquist (1961), Thomsen (1975) was able to construct exact tests for the variance components in an unbalanced two-way random model. Thomsen did this by first making an orthogonal transformation as outlined below. Let $\underset{\sim}{\bar{y}}$ be the vector of cell means, then in matrix notation we have (see Thomsen (1975, p. 258)):

$$\underset{\sim}{\bar{y}} = \mu \underset{\sim}{1}_{rs} + B_1 \underset{\sim}{\alpha} + B_2 \underset{\sim}{\beta} + I_{rs} \underset{\sim}{(\alpha\beta)} + \underset{\sim}{\bar{\epsilon}} \tag{2.3}$$

where $\underset{\sim}{1}_{rs}$ is the vector of ones of dimension rs; I_{rs} is the $rs \times rs$ identity matrix; B_1 and B_2 are defined as

$$\begin{aligned} B_1 &= I_r \otimes \underset{\sim}{1}_s \\ B_2 &= \underset{\sim}{1}_r \otimes I_s \end{aligned} \tag{2.4}$$

and r is the number of rows and s is the number of columns in the design. Since $A_1 = B_1 B_1'$ and $A_2 = B_2 B_2'$ commute, there exists an orthogonal matrix P which simultaneously diagonalizes A_1 and A_2. Further, the first row of P may be chosen to be $(1/\sqrt{rs})\underset{\sim}{1}_{rs}'$. Making the transformation

$$\underset{\sim}{z} = P\underset{\sim}{\bar{y}} \tag{2.5}$$

results in Thomsen's "semi-canonical" form. Thomsen used the last rs-1 elements of $\underset{\sim}{z}$ to derive exact tests for $\sigma^2_{\alpha\beta}$ (the variance component associated with the interaction effect) and for σ^2_α and σ^2_β (the main effects variance components). However, as in Spjøtvoll (1968), the development of the latter two tests rested on the assumption that $\sigma^2_{\alpha\beta} = 0$. Also, Thomsen assumed the layout contained no missing cells. (In a later section he dropped this assumption, but still assumed the design was connected.)

In 1983, Seely and El-Bassiouni returned to Wald's variance component test. In their paper, necessary and sufficient conditions were given under which the Wald test can be used in mixed models. They also developed a uniqueness property of Wald's test. Thus based solely on matrix ranks, one can determine if a particular variance component test is a Wald's test. They applied their notion to the tests developed by Spjøtvoll (1968) and Thomsen (1975). They concluded that the tests derived by Thomsen are indeed Wald's tests, while only the test for interaction developed by Spjøtvoll is a Wald's test.

Although both Spjøtvoll and Thomsen needed to assume additivity in their development of exact tests for the main effects variance components in an unbalanced random two-way model, Khuri and Littell (1987) demonstrated that this assumption was unnecessary. Initially, their work proceeded similarly to Thomsen (1975). However, they also made a particular transformation that reduced the analysis of the unbalanced model to that of a balanced one. Their idea was based on a scheme for resampling from the error vector. After making this transformation they were left with a model of the form (see Khuri and Littell (1987, p. 548)),

$$\underset{\sim}{\omega} = P_1 B_1 \underset{\sim}{\alpha} + P_1 B_2 \underset{\sim}{\beta} + \underset{\sim}{\epsilon}^* \qquad (2.6)$$

where $\underset{\sim}{\alpha} \sim N(\underset{\sim}{0}, \sigma_\alpha^2 I_r)$; $\underset{\sim}{\beta} \sim N(\underset{\sim}{0}, \sigma_\beta^2 I_s)$; $\underset{\sim}{\epsilon}^* \sim N(\underset{\sim}{0}, \delta I_{rs-1})$ where $\delta = \sigma_{\alpha\beta}^2 + \lambda_{\max}\, \sigma_\epsilon^2$ and $\lambda_{\max}$ is the largest eigenvalue of the matrix $P_1 K P_1'$ where $K = \mathrm{diag}(n_{11}^{-1}, n_{12}^{-1}, \cdots, n_{rs}^{-1})$ and P_1 consists of the last rs-1 rows of P (see (2.5)). Furthermore, $\underset{\sim}{\alpha}$, $\underset{\sim}{\beta}$ and $\underset{\sim}{\epsilon}^*$ are mutually independent. B_1 and B_2 are defined in (2.4).

After partitioning $\underset{\sim}{\omega}$ into $\underset{\sim}{\omega} = (\underset{\sim}{\omega}_\alpha', \underset{\sim}{\omega}_\beta', \underset{\sim}{\omega}_{\alpha\beta}')'$, Khuri and Littell developed exact tests for σ_α^2 and σ_β^2 based on $(\underset{\sim}{\omega}_\alpha', \underset{\sim}{\omega}_{\alpha\beta}')'$ and $(\underset{\sim}{\omega}_\beta', \underset{\sim}{\omega}_{\alpha\beta}')'$, respectively. In particular, their test statistic for testing $H_0{:}\sigma_\alpha^2 = 0$ is given by

$$F = \frac{\underset{\sim}{\omega}_\alpha' \underset{\sim}{\omega}_\alpha/(r-1)}{\underset{\sim}{\omega}_{\alpha\beta}' \underset{\sim}{\omega}_{\alpha\beta}/[(r-1)(s-1)]}. \qquad (2.7)$$

It should be noted that

$$\mathrm{rank}(P_1B_1 : P_1B_2) = r + s - 2$$

$$\mathrm{rank}(P_1B_1) = r - 1$$

$$\mathrm{rank}(P_1B_2) = s - 1.$$

Consequently, according to Seely and El-Bassiouni (1983), the tests derived by Khuri and Littell (1987) for the main effects variance components are Wald's tests based on the $\underset{\sim}{\omega}$ model in (2.6).

One attractive feature of a Wald's test is that when applied in an unbalanced one-way-random effects model, it may be interpreted as being "almost equal to the most powerful invariant tests against large alternatives" (Spjøtvoll (1967, p.422)). This result is also hinted at in Mostafa (1967). Also, under certain conditions, Mathew and Sinha (1988) show that the Wald test for testing the variance component associated with the unbalanced random two-way mixed model without interaction is the uniformly most powerful invariant test under a certain group of transformations. This is more fully discussed in chapter four.

Besides the above mentioned results, other exact procedures do exist in the literature. Khuri (1987) developed an exact test for the nesting effect's variance component in an unbalanced random 2-fold nested model. The idea of resampling from the error vector was also utilized in this paper. Verdooren (1988) also analyzed the unbalanced random 2-fold nested model. He was able to derive an exact test for ρ_1 for each given value of ρ_2. Here ρ_1 is the ratio of the nesting effect's variance component to the error variance component and ρ_2 is the ratio of the nested effect's variance component to the error variance component.

Exact tests for the random and fixed effects in some unbalanced mixed models were derived in Gallo (1987) and Gallo and Khuri (1990). Hocking (1988) also considered the mixed linear model. Although he was more interested in estimation than in testing, he did note that his estimators could be used to develop tests of hypotheses for designs with moderate imbalance. These tests, however, are not exact. We only mention them since they may represent reasonable alternatives to the methodology presented in Section three of Chapter three.

Turning to more general models, Khuri (1990) developed exact tests for random models with unequal cell frequencies in the last stage. These designs include all cross-classification random effects models with no missing cells. Utilizing resampling, Khuri was able to construct a set of mutually independent sums of squares that are distributed as scaled chi-squared variates. These sums of squares were then used to test hypotheses concerning the variance components. It should be noted that not all variance components can be tested exactly using Khuri's procedure. This should not be too surprising, however, since this phenomenon occurs even for some balanced random (or mixed) models (cf. Kendall and Stuart (1968), p. 83).

Using a different approach, Öfversten (1989) was also able to construct exact tests for the variance components in general mixed linear models. His idea was to use a result found in Allen (1974) to orthogonally transform the model matrices into so-called layer triangular form. The nice thing about this type of transformation is that when combined with orthogonal row permutations it can be used to reduce matrices to ones with full row rank. This technique is repeatedly utilized in Öfversten's work to produce exact tests for the variance components in certain mixed models. However, when certain rank conditions are not satisfied this approach cannot be used to produce the exact procedures. When this occurs, Öfversten relies on resampling to develop his exact tests. Öfversten's methodology can be used in any unbalanced mixed model to construct tests for the variance components. However, it is not altogether clear as to how to adapt his methods to designs not examined in his dissertation. More is said about Öfversten's results in Chapter four.

Partial motivation behind the need to develop exact testing procedures can be found in Tietjen (1974) and Cummings and Gaylor (1974). Both of these papers considered the random 2-fold nested model. In his simulation study, Tietjen showed that the approximate test based on Satterthwaite's (1946) procedure for testing the nesting effect's variance component can be markedly off in approximating the true p-value. This led him to recommend the use of the conventional F-test (that is, the test used when the data are balanced) as an approximate test for this hypothesis.

Cummings and Gaylor (1974) considered various 2-fold nested designs in their simulations. These designs were specifically chosen so that the mean squares involved in the approximate (Satterthwaite) F-test were either dependent, failed to be distributed as scaled chi-squared variates, or both. They demonstrated that when the mean squares were dependent, the estimated test size was less than the stated test size and thus, the approximate test was conservative. When the mean squares were independent but did not have chi-squared type distributions, the opposite occurred. The estimated test size was greater than the stated test size. When the mean squares were both dependent and not distributed as scaled chi-squared variates, a "counter-balancing" effect was observed. The approximate F-test in this case seemed to perform well. However, this counteracting effect may not occur in higher level designs. As Cummings and Gaylor (1974, p. 771) state "there is a possibility the test size disturbances due to mean square dependence and nonchisquaredness will provide some reinforcement."

In the next chapter we extend the work of Khuri (1987, 1990), Khuri and Littell (1987), and Gallo and Khuri (1990). The idea of resampling from the error vector is vital in our development and is used repeatedly throughout this dissertation. The reader is referred to Appendix C where the utility of this approach is illustrated.

CHAPTER THREE
EXACT TESTING PROCEDURES

3.1. The Unbalanced Random Two-Way Model with Some Cells Missing

3.1.1. Introduction

In this section we will consider the unbalanced random two-way model with some cells missing. Specifically, we will concentrate on the following model:

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk} \tag{3.1}$$

$\big((i,j) \in \mathcal{I};\ k = 1,2,\cdots,n_{ij}\big)$; where μ is an unknown constant parameter; α_i, β_j, $(\alpha\beta)_{ij}$, and ϵ_{ijk} are independently distributed normal random variables with zero means and variances σ^2_α, σ^2_β, $\sigma^2_{\alpha\beta}$, and σ^2_ϵ, respectively; $\mathcal{I}$ is the set of subscripts, (i,j), which label the nonempty cells of (3.1). That is,

$$\mathcal{I} = \{(i,j):\ \ 1 \le i \le r,\ 1 \le j \le s,\ n_{ij} > 0\} \tag{3.2}$$

where r is the number of rows and s the number of columns of the design. Note that $\mathcal{I}$ is a proper subset of IXJ, where $I = \{i : i = 1,2,\cdots,r\}$; $J = \{j : j = 1,2,\cdots,s\}$; and X denotes the Cartesian product.

In matrix form, (3.1) becomes

$$\underset{\sim}{y} = \mu \underset{\sim}{1}_n + X_1 \underset{\sim}{\alpha} + X_2 \underset{\sim}{\beta} + X_3 (\underset{\sim}{\alpha\beta}) + \underset{\sim}{\epsilon} \tag{3.3}$$

where $\underset{\sim}{y}$ is the vector of observations of dimension $n = \sum_{(i,j) \in \mathcal{I}} n_{ij}$; $\underset{\sim}{1}_n$ is a vector of ones of dimension n; X_1, X_2, and X_3 are known matrices of zeros and ones of orders $n \times r$, $n \times s$, and $n \times (rs - p)$, respectively, where p is the number of empty cells; $\underset{\sim}{\alpha}$, $\underset{\sim}{\beta}$, and $(\underset{\sim}{\alpha\beta})$ are column vectors whose elements are the α_i, β_j, and $(\alpha\beta)_{ij}$, respectively $\big((i,j) \in \mathcal{I})\big)$; $\underset{\sim}{\epsilon}$ is the $n \times 1$ vector of errors. Define

$$\bar{y}_{ij} = \frac{1}{n_{ij}} \sum_{k=1}^{n_{ij}} y_{ijk} \tag{3.4}$$

for all $(i,j) \in \mathcal{I}$. From (3.1) we obtain

$$\bar{y}_{ij} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \bar{\epsilon}_{ij} \tag{3.5}$$

$\big((i,j) \in \mathcal{I}\big)$, where

$$\bar{\epsilon}_{ij} = \frac{1}{n_{ij}} \sum_{k=1}^{n_{ij}} \epsilon_{ijk} \tag{3.6}$$

for all $(i,j) \in \mathcal{I}$. In matrix form, (3.5) becomes

$$\underset{\sim}{\bar{y}} = \mu \underset{\sim}{1}_{rs-p} + A_1 \underset{\sim}{\alpha} + A_2 \underset{\sim}{\beta} + I_{rs-p} (\underset{\sim}{\alpha\beta}) + \underset{\sim}{\bar{\epsilon}} \tag{3.7}$$

where $\underset{\sim}{\bar{y}}$ is the $(rs\text{-}p) \times 1$ vector of (filled) cell means; I_{rs-p} is the identity matrix of order rs-p; A_1 and A_2 are known matrices of zeros and ones of orders $(rs-p)\times r$ and $(rs-p)\times s$, respectively. A_1 and A_2 can be expressed as

$$A_1 = \overset{r}{\underset{i=1}{\oplus}} 1_{s-p_i} \tag{3.8}$$

$$A_2 = \begin{bmatrix} (I_{s-p_1} : 0_{(s-p_1)\times p_1})E_1 \\ (I_{s-p_2} : 0_{(s-p_2)\times p_2})E_2 \\ \vdots \\ (I_{s-p_r} : 0_{(s-p_r)\times p_r})E_r \end{bmatrix} \tag{3.9}$$

where p_i is the number of empty cells in row i $(i=1,2,\cdots,r)$; $0_{(s-p_i)\times p_i}$ is a matrix of zeros of order $(s-p_i)\times p_i$ $(i=1,2,\cdots,r)$; and E_i is a product of interchange matrices $(i=1,2,\cdots,r)$. We take $(I_{s-p_i} : 0_{(s-p_i)\times p_i}) = I_s$ when $p_i = 0$ $(i=1,2,\cdots,r)$.

Example 3.1

Consider the following two-way table, an “X” represents a filled cell.

X	X	X	X
X		X	X
X	X		

Then,

$$r = 3$$

$$s = 4$$

$$p_1 = 0$$

$$p_2 = 1$$

$$p_3 = 2$$

$$p = 3$$

furthermore,

$$E_1 = I_4 = E_3$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

consequently,

$$A_1 = \mathrm{diag}(\underset{\sim}{1}_4, \underset{\sim}{1}_3, \underset{\sim}{1}_2)$$

$$A_2 = \left[\begin{array}{c} I_4 \\ \hdashline \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ \hdashline (I_2 : 0_{2\times 2}) \end{array}\right].$$

Besides the usual assumptions concerning the random effects, we will also assume:

(i) The model used is model (3.1) (or model (3.3))

(ii) Both r and s are at least two

(iii) $\operatorname{rank}(A_1 : A_2) = r + s - 1$ (3.10)

(iv) $p < (r-1)(s-1)$

(v) $n > 2(rs-p) - \min(r,s)$

Assumption (iii) implies that the design is connected (Gately (1962) p. 46), while the last assumption (along with the others) is needed to insure that the exact tests for the main effects' variance components are valid.

The inclusion of missing cells into any design adds a degree of difficulty to the analysis. The two-way random model is no exception. One problem is that the matrices $B_1 = A_1A_1'$ and $B_2 = A_2A_2'$ do not, in general, commute. B_1 and B_2 will commute if all cells are filled (see Thomsen (1975) or Khuri and Littell (1987)).

Reconsidering Example 3.1, we have

$$B_1 = \mathrm{diag}(J_4,\ J_3,\ J_2)$$

where $J_a = \underset{\sim}{1}_a \underset{\sim}{1}'_a$. While

$$B_2 = \begin{bmatrix} I_4 & E'_2 \begin{pmatrix} I_3 \\ : \\ \underset{\sim}{0}' \end{pmatrix} & \begin{pmatrix} I_2 \\ : \\ 0 \end{pmatrix} \\ (I_3 : \underset{\sim}{0}) E_2 & I_3 & (I_3 : \underset{\sim}{0})\ E_2 \begin{pmatrix} I_2 \\ : \\ 0 \end{pmatrix} \\ (I_2 : 0) & (I_2 : 0) E'_2 \begin{pmatrix} I_3 \\ : \\ \underset{\sim}{0}' \end{pmatrix} & I_2 \end{bmatrix}$$

hence

$$B_1 B_2 = \begin{bmatrix} J_4 & J_4 E'_2 \begin{pmatrix} I_3 \\ : \\ \underset{\sim}{0}' \end{pmatrix} & J_4 \begin{pmatrix} I_2 \\ : \\ 0 \end{pmatrix} \\ J_3 (I_3 : \underset{\sim}{0}) E_2 & J_3 & J_3 (I_3 : \underset{\sim}{0}) E_2 \begin{pmatrix} I_2 \\ : \\ 0 \end{pmatrix} \\ J_2 (I_2 : 0) & J_2 (I_2 : 0) E'_2 \begin{pmatrix} I_3 \\ : \\ \underset{\sim}{0}' \end{pmatrix} & J_2 \end{bmatrix}.$$

Since, for example, $J_4 \begin{pmatrix} I_2 \\ : \\ 0 \end{pmatrix} = \begin{pmatrix} J_2 \\ J_2 \end{pmatrix}$ while $J_2(I_2 : 0) = (J_2 : 0)$, we see that $B_1 B_2$ is not symmetric. Thus B_1 and B_2 do not commute. Consequently no orthogonal matrix exists that will simultaneously diagonalize B_1 and B_2.

The next section lays the groundwork necessary for the construction of the exact tests.

3.1.2 Preliminary Development

Recall (3.7). From the distributional properties of the random effects, we can conclude that

$$\underset{\sim}{\bar{y}} \sim N(\mu \underset{\sim}{1}_{rs-p}, \Sigma)$$

where

$$\Sigma = B_1\sigma_\alpha^2 + B_2\sigma_\beta^2 + I_{rs-p}\sigma_{\alpha\beta}^2 + K\sigma_\epsilon^2 \tag{3.11}$$

and

$$K = \bigoplus_{(i,j)\in\mathscr{S}} n_{ij}^{-1} \tag{3.12}$$

$$\begin{aligned} B_1 &= A_1A_1' \\ B_2 &= A_2A_2' \end{aligned} \tag{3.13}$$

The following lemma provides the ranks of B_1 and B_2.

Lemma 3.1

(a) $\mathrm{rank}(B_1) = r$

(b) $\mathrm{rank}(B_2) = s$

proof:

(a) $\mathrm{rank}(B_1) = \mathrm{rank}(A_1A_1') = \mathrm{rank}(A_1)$. Since

$$\mathrm{rank}(A_1) = (\overset{r}{\underset{i=1}{\oplus}} 1_{s-p_i}) \qquad \bigl(\text{see } (3.8)\bigr)$$

$$= \sum_{i=1}^{r} (1) = r, \text{ the result follows.}$$

(b) Without loss of generality, we can assume that each column (and row) in the layout includes at least one cell containing observations. Thus, each column in A_2 contains at least one "1". Hence there exists a permutation matrix Π such that

$$\Pi A_2 = \begin{bmatrix} I_s \\ \cdot\cdot \\ \Phi \end{bmatrix}$$

for some $(rs-p-s)\times s$ matrix Φ. Consequently,

$$\mathrm{rank}(B_2) = \mathrm{rank}(A_2A_2')$$

$$= \mathrm{rank}(A_2)$$

$$= \text{rank}(\Pi A_2)$$

$$= \text{rank}\begin{bmatrix} I_s \\ \cdot\cdot \\ \Phi \end{bmatrix} = s. \qquad \square$$

Now, let

$$\underset{\sim}{u} = H_1\bar{\underset{\sim}{y}} \tag{3.14}$$

where H_1 consists of the last $rs-p-1$ rows of the $(rs-p)\times(rs-p)$ Helmert matrix (see Searle (1982), p. 71). We note that $H_1\underset{\sim}{1}_{rs\text{-}p} = \underset{\sim}{0}_{rs\text{-}p-1}$. Then, it is easily verified that

$$E(\underset{\sim}{u}) = \underset{\sim}{0}_{rs\text{-}p\text{-}1}$$

$$\text{Var}(\underset{\sim}{u}) = H_1B_1H_1'\sigma_\alpha^2 + H_1B_2H_1'\sigma_\beta^2 + I_{rs\text{-}p\text{-}1}\sigma_{\alpha\beta}^2 + H_1KH_1'\sigma_\epsilon^2. \tag{3.15}$$

The purpose of this transformation is to construct a vector which has a zero mean vector. The ranks of $H_1B_1H_1'$ and $H_1B_2H_1'$ are examined in the next lemma.

Lemma 3.2

(a) $\text{rank}(H_1B_1H_1') = r-1$

(b) $\operatorname{rank}(H_1 B_2 H_1') = s - 1$

(c) $\operatorname{rank}(H_1 B_1 H_1' + H_1 B_2 H_1') = r + s - 2$

Proof:

(a) $\operatorname{rank}(H_1 B_1 H_1') = r(H_1 A_1 A_1' H_1')$

$= r(H_1 A_1)$

$\leq r(A_1) = r$ (from Lemma 3.1(a)).

However,

$$A_1 \underset{\sim}{1}_r = \left(\bigoplus_{i=1}^{r} \underset{\sim}{1}_{s-p_i}\right) \underset{\sim}{1}_r = \underset{\sim}{1}_{rs-p}.$$

Thus,

$$H_1 A_1 \underset{\sim}{1}_r = H_1 \underset{\sim}{1}_{rs-p} = \underset{\sim}{0}_{rs-p-1}.$$

Therefore,

$$\operatorname{rank}(H_1 B_1 H_1') \leq r - 1.$$

Conversely, by the Frobenius Inequality (see Lemma A.1), we have

$$\begin{aligned}\operatorname{rank}(H_1B_1H_1') &= \operatorname{rank}(H_1A_1)\\ &\geq \operatorname{rank}(H_1)+\operatorname{rank}(A_1)-\operatorname{rank}(I_{rs-p})\\ &= rs-p-1+r-(rs-p)=r-1.\end{aligned}$$

Consequently, $\operatorname{rank}(H_1B_1H_1')=r-1$.

(b) Similar to (a).

(c) $\operatorname{rank}(H_1B_1H_1'+H_1B_2H_1') \leq \operatorname{rank}(H_1B_1H_1')+\operatorname{rank}(H_1B_2H_1')$

$$= r+s-2 \quad \Big(\text{from (a) and (b)}\Big).$$

Conversely, the Frobenius Inequality gives us

$$\begin{aligned}\operatorname{rank}(H_1B_1H_1'+H_1B_2H_1') &= \operatorname{rank}\Big(H_1(B_1+B_2)H_1'\Big)\\ &= \operatorname{rank}\Big(H_1(A_1A_1'+A_2A_2')H_1'\Big)\\ &= \operatorname{rank}\left(H_1[A_1:A_2]\begin{bmatrix}A_1'\\ \cdot\cdot\\ A_2'\end{bmatrix}H_1'\right)\\ &= \operatorname{rank}\Big(H_1[A_1:A_2]\Big)\\ &\geq \operatorname{rank}(H_1)+\operatorname{rank}\Big([A_1:A_2]\Big)-\operatorname{rank}(I_{rs-p})\end{aligned}$$

$$= rs - p - 1 + r + s - 1 - (rs - p)$$

$$= r + s - 2$$

$\left(\text{by assumption, rank } \left([A_1 : A_2]\right) = r + s - 1 \left(\text{see (3.10)}\right)\right)$. □

The next step in our analysis is to simultaneously diagonalize $H_1B_1H_1'$ and $H_1B_2H_1'$. Because B_1 and B_2 do not, in general, commute, neither will $H_1B_1H_1'$ and $H_1B_2H_1'$. So we cannot achieve simultaneous diagonalization by way of an orthogonal transformation. However, both $H_1B_1H_1'$ and $H_1B_2H_1'$ are n.n.d. Consequently a nonsingular matrix, M, of order $(rs - p - 1) \times (rs - p - 1)$, will exist that will diagonalize both $H_1B_1H_1'$ and $H_1B_2H_1'$ (see Lemma A.4). Define

$$\Delta_1 = MH_1B_1H_1'M' \tag{3.16}$$

$$\Delta_2 = MH_1B_2H_1'M' \tag{3.17}$$

Both Δ_1 and Δ_2 are diagonal matrices of order $rs - p - 1$. By Corollary A.4.1 of Lemma A, we can express Δ_1 and Δ_2 as

$$\Delta_1 = \operatorname{diag}\left(I_{r-1},\ 0_{s-1},\ 0_{(r-1)(s-1)-p}\right) \tag{3.18}$$

$$\Delta_2 = \operatorname{diag}\left(0_{r-1},\ I_{s-1},\ 0_{(r-1)(s-1)-p}\right). \tag{3.19}$$

Consider, then, the $(rs\text{-}p\text{-}1) \times 1$ vector $\underset{\sim}{v}$ defined as

$$\underset{\sim}{v} = M\underset{\sim}{u} . \tag{3.20}$$

It is easily shown that

$$E(\underset{\sim}{v}) = \underset{\sim}{0} \tag{3.21}$$

$$\mathrm{Var}(\underset{\sim}{v}) = \Delta_1\sigma^2_\alpha + \Delta_2\sigma^2_\beta + F\sigma^2_{\alpha\beta} + G\sigma^2_\epsilon$$

where

$$\begin{aligned} F &= MM' \\ G &= MH_1KH_1'M' . \end{aligned} \tag{3.22}$$

We can partition $\underset{\sim}{v}$ into

$$\underset{\sim}{v} = (\underset{\sim}{v}_1',\ \underset{\sim}{v}_2',\ \underset{\sim}{v}_3')' , \tag{3.23}$$

where $\underset{\sim}{v}_1$ consists of the $r-1$ elements of $\underset{\sim}{v}$ whose covariance matrix is free from σ^2_β; $\underset{\sim}{v}_2$ consists of the $s-1$ elements of $\underset{\sim}{v}$ whose covariance matrix is free from σ^2_α; and $\underset{\sim}{v}_3$ consists of the $(r-1)(s-1)-p$ elements of $\underset{\sim}{v}$ whose covariance matrix is free from both σ^2_α and σ^2_β. Hence,

$$E(\underset{\sim}{v}) = \underset{\sim}{0}$$

$$\mathrm{Var}(\underset{\sim}{v}) = \mathrm{diag}(I_{r-1}\sigma_\alpha^2,\ I_{s-1}\sigma_\beta^2,\ 0) + F\sigma_{\alpha\beta}^2 + G\sigma_\epsilon^2. \tag{3.24}$$

This result follows from (3.18), (3.19), and (3.21).

We are now ready to develop our exact tests. To begin with, we develop an exact test concerning σ_α^2.

3.1.3 An Exact Test Concerning σ_α^2

Initial development. In this section an exact test for testing $H_0:\sigma_\alpha^2 = 0$ vs. $H_a:\sigma_\alpha^2 \neq 0$ is derived by utilizing the subvectors $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_3$ of $\underset{\sim}{v}$ in (3.23). To begin with, partition F and G $\big($see (3.22)$\big)$ as

$$F = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{12}' & F_{22} & F_{23} \\ F_{13}' & F_{23}' & F_{33} \end{bmatrix} \tag{3.25}$$

and

$$G = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{12}' & G_{22} & G_{23} \\ G_{13}' & G_{23}' & G_{33} \end{bmatrix} \tag{3.26}$$

where $F_{11}(G_{11})$, $F_{22}(G_{22})$, and $F_{33}(G_{33})$ are, respectively, $(r-1)\times(r-1)$, $(s-1)\times(s-1)$, and $\big((r-1)(s-1)-p\big)\times\big((r-1)(s-1)-p\big)$ matrices. Also, let $\underset{\sim}{v}_{1:3}$ be the $\big(s(r-1)-p\big)\times 1$ vector defined as

$$\underset{\sim}{v}_{1:3} = \begin{bmatrix} \underset{\sim}{v}_1 \\ \underset{\sim}{v}_3 \end{bmatrix} \tag{3.27}$$

where $\underset{\sim}{v}_1$ is $(r-1)\times 1$ while $\underset{\sim}{v}_3$ is $\big((r-1(s-1)-p\big)\times 1$. Then, from (3.21) we have

$$E(\underset{\sim}{v}_{1:3}) = \underset{\sim}{0} \tag{3.28}$$

$$\mathrm{Var}(\underset{\sim}{v}_{1:3}) = \Delta_1\sigma_\alpha^2 + F_{1:3}\sigma_{\alpha\beta}^2 + G_{1:3}\sigma_\epsilon^2$$

where

$$\Delta_1 = \begin{bmatrix} I_{r-1} & 0 \\ 0 & 0_{(r-1)(s-1)-p} \end{bmatrix} \tag{3.29}$$

$$F_{1:3} = \begin{bmatrix} F_{11} & F_{13} \\ F'_{13} & F_{33} \end{bmatrix} \tag{3.30}$$

$$G_{1:3} = \begin{bmatrix} G_{11} & G_{13} \\ G'_{13} & G_{33} \end{bmatrix}. \tag{3.31}$$

From (3.22) notice that since both F and G are positive definite (p.d.), $F_{1:3}$ and $G_{1:3}$ are also both p.d.

Now, there exists a nonsingular matrix N or order $s(r-1)-p$ such that (see Lemma A.3)

$$NF_{1:3}N' = I_{s(r-1)-p} \tag{3.32}$$

and

$$N\Delta_1 N' = \Lambda_1, \tag{3.33}$$

where Λ_1 is a diagonal matrix whose diagonal entries are solutions θ to

$$|\Delta_1 - \theta F_{1:3}| = 0. \tag{3.34}$$

We note that if $F_{1:3}^{11}$ represents the first $r-1$ rows and columns of $F_{1:3}^{-1}$ then the nonzero solutions to (3.34) are exactly the eigenvalues of $F_{1:3}^{11}$.

Without loss of generality we may express Λ_1 in (3.33) as

$$\Lambda_1 = \begin{bmatrix} \Lambda_1^* & 0 \\ 0 & 0 \end{bmatrix} \tag{3.35}$$

where Λ_1^* is a diagonal matrix of order $r-1$ whose diagonal entries are the eigenvalues of $F_{1:3}^{11}$. (If Λ_1 is not in this form then we can find an orthogonal matrix P such that

$$P\Lambda_1 P' = \begin{bmatrix} \Lambda_1^* & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, letting

$$N^* = PN$$

we see that

$$\begin{aligned} N^*\Delta_1 N^{*\prime} &= PN\Delta_1 N'P' \\ &= P\Lambda_1 P' \qquad \Big(\text{see } (3.33)\Big) \\ &= \begin{bmatrix} \Lambda_1^* & 0 \\ 0 & 0 \end{bmatrix}.\Big) \end{aligned}$$

The purpose of introducing the matrix N was to simultaneously diagonalize Δ_1 and $F_{1:3}$ in the expression for $\mathrm{Var}(\underset{\sim}{v}_{1:3})$ given in (3.28).

Define the $s(r-1)-p$ random vector $\underset{\sim}{x}_{1:3}$ as

$$\underset{\sim}{x}_{1:3} = \begin{bmatrix} \underset{\sim}{x}_1 \\ \underset{\sim}{x}_3 \end{bmatrix} = N\begin{bmatrix} \underset{\sim}{v}_1 \\ \underset{\sim}{v}_3 \end{bmatrix} = N\underset{\sim}{v}_{1:3} \tag{3.36}$$

where $\underset{\sim}{x}_1$ is $(r-1)\times 1$ and $\underset{\sim}{x}_3$ is $\Big((r-1)(s-1)-p\Big)\times 1$. Then

$$\begin{aligned} E(\underset{\sim}{x}_{1:3}) &= \underset{\sim}{0} \\ \mathrm{Var}(\underset{\sim}{x}_{1:3}) &= \Lambda_1\sigma_\alpha^2 + I_{s(r-1)-p}\sigma_{\alpha\beta}^2 + L_1\sigma_\epsilon^2. \end{aligned} \tag{3.37}$$

This follows from (3.28), (3.32), and (3.33). L_1 is the $\big(s(r-1)-p\big) \times \big(s(r-1)-p\big)$ full rank matrix

$$L_1 = NG_{1:3}N'. \tag{3.38}$$

We can equivalently express $\mathrm{Var}(\underset{\sim}{x}_{1:3})$ in (3.37) as

$$\mathrm{Var}(\underset{\sim}{x}_{1:3}) = \mathrm{diag}\Big(\Lambda_1^*\sigma_\alpha^2 + I_{r-1}\sigma_{\alpha\beta}^2,\ I_{(r-1)(s-1)-p}\sigma_{\alpha\beta}^2\Big) + L_1\sigma_\epsilon^2. \tag{3.39}$$

Because L_1 is not diagonal, $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_3$ are not independent. To achieve independence between $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_3$, we resample from the error vector. The results of Lemma B.6 are utilized heavily in the forthcoming development.

Utilizing resampling. Because model (3.1) or (3.3) is a special case of model (B.1) the error sum of squares, SSE, associated with this model can be expressed as

$$SSE = \underset{\sim}{y}'R\underset{\sim}{y} \tag{3.40}$$

where R is the n×n symmetric, idempotent matrix

$$R = I_n - W(W'W)^-W' \tag{3.41}$$

where

$$W = [\underset{\sim}{1}_n : X_1 : X_2 : X_3] \tag{3.42}$$

and X_1, X_2, and X_3 are described in (3.3). Since

$$\text{range}(\underset{\sim}{1}_n) \subset \text{range}(X_3)$$

$$\text{range}(X_i) \subset \text{range}(X_3) \qquad (i = 1,2),$$

(3.41) can also be written as

$$\begin{aligned} R &= I_n - X_3(X_3'X_3)^{-1}X_3' \\ &= I_n - \bigoplus_{(i,j) \in \mathcal{I}} (J_{n_{ij}}/n_{ij}). \end{aligned} \tag{3.43}$$

This last equality follows since

$$X_3 = \bigoplus_{(i,j) \in \mathcal{I}} \underset{\sim}{1}_{n_{ij}}. \tag{3.44}$$

Following Lemma B.6 (iv), there exists an orthogonal matrix C and a diagonal matrix Γ such that

$$R = C\Gamma C'. \tag{3.45}$$

Partition C and Γ as

$$C = [C_1 : C_2 : C_3] \tag{3.46}$$

$$\Gamma = \mathrm{diag}(I_{\eta_1}, I_{\eta_2}, 0)$$

where

$$\begin{aligned} \eta_1 &= s(r-1)-p \\ \eta_2 &= n-2(rs-p)+s. \end{aligned} \tag{3.47}$$

We note that $\eta_1+\eta_2 = n-(rs-p) = \mathrm{rank}(R)$ and $\eta_2 > 0$ by (3.10)(v). C_1, C_2, and C_3 are matrices or orders $n \times \eta_1$, $n \times \eta_2$ and $n \times (rs-p)$, respectively. From (3.46), R can be written as

$$R = C_1 C_1' + C_2 C_2'. \tag{3.48}$$

Now, consider the $\big(s(r-1)-p\big) \times 1$ random vector $\underset{\sim}{\omega}_{1:3}$ defined to be

$$\underset{\sim}{\omega}_{1:3} = \underset{\sim}{x}_{1:3} + (\lambda_{1,\max} I_{\eta_1} - L_1)^{\frac{1}{2}} C_1' \underset{\sim}{y} \tag{3.49}$$

where $\lambda_{1,\max}$ is the largest eigenvalue of L_1 and $(\lambda_{1,\max} I_{\eta_1} - L_1)^{\frac{1}{2}}$ is a symmetric matrix with eigenvalues equal to the square roots of the eigenvalues of $(\lambda_{1,\max} I_{\eta_1} - L_1)$, which are nonnegative.

Let $\underset{\sim}{\omega}_{1:3}$ be partitioned just like $\underset{\sim}{x}_{1:3}$ in (3.36), that is

$$\underset{\sim}{\omega}_{1:3} = \begin{pmatrix} \underset{\sim}{\omega}_1 \\ \underset{\sim}{\omega}_3 \end{pmatrix} \tag{3.50}$$

where $\underset{\sim}{\omega}_1$ and $\underset{\sim}{\omega}_3$ are of orders $r-1$ and $(r-1)(s-1)-p$, respectively.

The major result of this section is given in the following theorem.

Theorem 3.1

(a) $E(\underset{\sim}{\omega}_1) = \underset{\sim}{0}$; $E(\underset{\sim}{\omega}_3) = \underset{\sim}{0}$

(b) $\underset{\sim}{\omega}_1$ and $\underset{\sim}{\omega}_3$ are independent, normally distributed random vectors and have the following variance-covariance matrices:

$$\begin{aligned} \operatorname{Var}(\underset{\sim}{\omega}_1) &= \Lambda_1^* \sigma_\alpha^2 + \delta_1 I_{r-1} \\ \operatorname{Var}(\underset{\sim}{\omega}_3) &= \delta_1 I_{(r-1)(s-1)-p} \end{aligned} \tag{3.51}$$

where $\delta_1 = \sigma_{\alpha\beta}^2 + \lambda_{1,\max}\ \sigma_\epsilon^2$.

Proof:

(a) From (3.49) we have

$$E(\underset{\sim}{\omega}_{1:3}) = E(\underset{\sim}{x}_{1:3}) + (\lambda_{1,\max} I_{\eta_1} - L_1)^{\frac{1}{2}} C_1' E(\underset{\sim}{y})$$

From (3.37) we know that $E(\underset{\sim}{x}_{1:3}) = \underset{\sim}{0}$. Since $R\underset{\sim}{1}_n = \underset{\sim}{0}$ $\big($see Lemma B.6(ii)$\big)$ and $\operatorname{range}(C_1) \subset \operatorname{range}(R)$, we have that $C_1'\underset{\sim}{1}_n = \underset{\sim}{0}$. Thus the result follows after noting that $E(\underset{\sim}{y}) = \mu\underset{\sim}{1}_n$ $\big($see (3.3)$\big)$.

(b) Clearly $\underset{\sim}{\omega}_1$ and $\underset{\sim}{\omega}_3$ are normally distributed. From Lemma B.6(v) we have that $\bar{\underset{\sim}{y}}$ is independent of SSE. Consequently

$$D\Omega R = 0$$

where D is such that $D\underset{\sim}{y} = \bar{\underset{\sim}{y}}$, and

$$\Omega = \mathrm{Var}(\underset{\sim}{y}) = X_1X_1'\sigma_\alpha^2 + X_2X_2'\sigma_\beta^2 + X_3X_3'\sigma_{\alpha\beta}^2 + \sigma_\epsilon^2 I_n \qquad (3.52)$$

$\Big($see (3.3)$\Big)$. However, since $\mathrm{range}(C_1) \subset \mathrm{range}(R)$ it follows that

$$D\Omega C_1 = 0.$$

Thus $\bar{\underset{\sim}{y}}$ is independent of $C_1'\underset{\sim}{y}$. Since $\underset{\sim}{x}_{1:3}$ is a function of $\bar{\underset{\sim}{y}}$, it must also be independent of $C_1'\underset{\sim}{y}$. Hence

$$\mathrm{Var}(\underset{\sim}{\omega}_{1:3}) = \mathrm{Var}(\underset{\sim}{x}_{1:3}) + (\lambda_{1,\max} I_{\eta_1} - L_1)^{\frac{1}{2}} C_1'\Omega C_1 (\lambda_{1,\max} I_{\eta_1} - L_1)^{\frac{1}{2}}. \qquad (3.53)$$

From Lemma B.6(ii), $R\Omega R = R^2\sigma_\epsilon^2$. Thus since $\mathrm{range}(C_1) \subset \mathrm{range}(R)$ and R is idempotent we have

$$C_1'\Omega C_1 = C_1'C_1\sigma_\epsilon^2 = \sigma_\epsilon^2 I_{\eta_1}.$$

The last equality follows from the fact that $C = [C_1 : C_2 : C_3]$ is orthogonal. Therefore, (3.53) can be expressed as

$$\begin{aligned}\mathrm{Var}(\underset{\sim}{\omega}_{1:3}) &= \mathrm{Var}(\underset{\sim}{x}_{1:3}) + (\lambda_{1,\max}\mathrm{I}_{\eta_1} - \mathrm{L}_1)\sigma_\epsilon^2 \\ &= \Lambda_1\sigma_\alpha^2 + \mathrm{I}_{s(r-1)-p}\sigma_{\alpha\beta}^2 + \mathrm{L}_1\sigma_\epsilon^2 + (\lambda_{1,\max}\mathrm{I}_{\eta_1} - \mathrm{L}_1)\sigma_\epsilon^2 \\ &= \Lambda_1\sigma_\alpha^2 + \mathrm{I}_{s(r-1)-p}(\sigma_{\alpha\beta}^2 + \lambda_{1,\max}\sigma_\epsilon^2)\,.\end{aligned}$$

Consequently, from (3.35),

$$\mathrm{Var}(\underset{\sim}{\omega}_1) = \Lambda_1^*\sigma_\alpha^2 + \delta_1\mathrm{I}_{r-1}$$

$$\mathrm{Var}(\underset{\sim}{\omega}_3) = \delta_1\mathrm{I}_{(r-1)(s-1)-p}$$

where $\delta_1 = \sigma_{\alpha\beta}^2 + \lambda_{1,\max}\sigma_\epsilon^2$. □

As a result of Theorem 3.1 we obtain

(i) $\mathrm{SS}_1 = \underset{\sim}{\omega}_1'(\Lambda_1^*\sigma_\alpha^2 + \delta_1\mathrm{I}_{r-1})^{-1}\underset{\sim}{\omega}_1 \sim \chi_{r-1}^2$ (3.54)

(ii) $\mathrm{SS}_3 = \underset{\sim}{\omega}_3'\underset{\sim}{\omega}_3/\delta_1 \sim \chi_{(r-1)(s-1)-p}^2$ (3.55)

(iii) SS_1 and SS_3 are independent.

A test statistic for testing $\mathrm{H}_0:\sigma_\alpha^2 = 0$ vs. $\mathrm{H}_a:\sigma_\alpha^2 \neq 0$ is then

$$\mathrm{F}_1 = \frac{\underset{\sim}{\omega}_1'\underset{\sim}{\omega}_1/(r-1)}{\underset{\sim}{\omega}_3'\underset{\sim}{\omega}_3/\big((r-1)(s-1)-p\big)}\ . \tag{3.56}$$

Under H_0, $F_1 \sim F_{r-1,(r-1)(s-1)-p}$. Note that

$$E\left(\frac{\underset{\sim}{\omega}_1'\underset{\sim}{\omega}_1}{r-1}\right) = \bar{\theta}_1\sigma_\alpha^2 + \delta_1 \tag{3.57}$$

where $\bar{\theta}_1$ is the average of the eigenvalues of $F_{1:3}^{11}$ (see (3.35)). Since $F_{1:3}$ (see (3.30)) is p.d. it follows that $F_{1:3}^{-1}$ is p.d. Thus $F_{1:3}^{11}$ is p.d., implying that $\bar{\theta}_1 > 0$. Consequently, we would reject H_0 at the γ-level of significance if

$$F_1 > F_{\gamma,r-1,(r-1)(s-1)-p}$$

where $F_{\gamma,r-1,(r-1)(s-1)-p}$ is the upper 100γ% point of the corresponding F-distribution.

We now turn our attention to the construction of an exact test concerning σ_β^2.

3.1.4 An Exact Test Concerning σ_β^2

Initial development. This section is concerned with the construction of an exact test for σ_β^2. Much of the development in this section parallels that of Section 3.1.4 except that we now focus on the subvectors $\underset{\sim}{v}_2$ and $\underset{\sim}{v}_3$ of $\underset{\sim}{v}$ in (3.23).

Let $\underset{\sim}{v}_{2:3}$ be the $\left(r(s-1)-p\right)\times 1$ random vector

$$\underset{\sim}{v}_{2:3} = \begin{bmatrix} \underset{\sim}{v}_2 \\ \underset{\sim}{v}_3 \end{bmatrix} \tag{3.58}$$

where $\underset{\sim}{y}_2$ is $(s-1)\times 1$ and $\underset{\sim}{y}_3$ (recall) is $\big((r-1)(s-1)-p\big)\times 1$. Similarly to Section 3.1.3 we can make a nonsingular transformation, T, to obtain a random vector $\underset{\sim}{x}_{2:3} = [\underset{\sim}{x}_2' : \underset{\sim}{x}_3^{*\prime}]' = T\underset{\sim}{y}_{2:3}$ with

$$\begin{aligned} E(\underset{\sim}{x}_{2:3}) &= \underset{\sim}{0} \\ \operatorname{Var}(\underset{\sim}{x}_{2:3}) &= \Lambda_2\sigma_\beta^2 + I_{r(s-1)-p}\sigma_{\alpha\beta}^2 + L_2\sigma_\epsilon^2 \end{aligned} \tag{3.59}$$

where $\Lambda_2 = \operatorname{diag}(\Lambda_2^*,\ 0)$ and Λ_2^* is a diagonal matrix of order s-1 whose diagonal entries are the eigenvalues of $F_{2:3}^{11}$. $F_{2:3}^{11}$ represents the first s-1 rows and columns of $F_{2:3}^{-1}$ where

$$F_{2:3} = \begin{bmatrix} F_{22} & F_{23} \\ F_{23}' & F_{33} \end{bmatrix} \tag{3.60}$$

$\big($see (3.25)$\big)$. Also, L_2 is the $\big(r(s-1)-p\big)\times\big(r(s-1)-p\big)$ full rank matrix

$$L_2 = TG_{2:3}T'. \tag{3.61}$$

T is the nonsingular matrix that simultaneously diagonalizes $F_{2:3}$ and Δ_2 where

$$\Delta_2 = \begin{bmatrix} I_{s-1} & 0 \\ 0 & 0_{(r-1)(s-1)-p} \end{bmatrix} \tag{3.62}$$

$\big($see(3.24)$\big)$; and $G_{2:3}$ is defined to be

$$G_{2:3} = \begin{bmatrix} G_{22} & G_{22} \\ G_{23}' & G_{23} \end{bmatrix} \tag{3.63}$$

$\bigl($see (3.26)$\bigr)$.

We note that $\underset{\sim}{x}_3^*$, defined in $\underset{\sim}{x}_{2:3}$ above, is not the same as $\underset{\sim}{x}_3$ in (3.36) since a different nonsingular transformation was necessary to simultaneously diagonalize $F_{2:3}$ and Δ_2.

Because $\underset{\sim}{x}_2$ and $\underset{\sim}{x}_3^*$ are not independent $\bigl(L_2$ is not diagonal (see (3.39))$\bigr)$, we will again need to employ resampling to aid in the construction of the exact test.

Utilizing resampling. Consider again the matrix R defined in (3.41) or (3.43). Unlike the partitioning of C and Γ described in (3.46), we now partition C and Γ as

$$\begin{aligned} C &= [C_1^* : C_2^* : C_3^*] \\ \Gamma &= \mathrm{diag}(I_{\eta_1^*}, I_{\eta_2^*}, 0) \end{aligned} \tag{3.64}$$

where

$$\begin{aligned} \eta_1^* &= r(s-1)-p \\ \eta_2^* &= n-2(rs-p)+r. \end{aligned} \tag{3.65}$$

Note that $\eta_1^* + \eta_2^* = n-(rs-p) = \mathrm{rank}(R)$ and $\eta_2^* > 0$ by (3.10)(v). C_1^*; C_2^*, and C_3^* are matrices of orders $n\times\eta_1^*$, $n\times\eta_2^*$, and $n\times(rs-p)$, respectively. From (3.64), we can express R as

$$R = C_1^* C_1^{*\prime} + C_2^* C_2^{*\prime} \tag{3.66}$$

(see also (3.45)).

Now define the $(r(s-1)-p)\times 1$ random vector $\underset{\sim}{\omega}_{2:3}$ as

$$\underset{\sim}{\omega}_{2:3} = \underset{\sim}{x}_{2:3} + (\lambda_{2,\max} I_{\eta_1^*} - L_2)^{\frac{1}{2}} C_1^{*\prime} \underset{\sim}{y} \tag{3.67}$$

where $\lambda_{2,\max}$ is the largest eigenvalue of L_2 and $(\lambda_{2,\max} I_{\eta_1^*} - L_2)^{\frac{1}{2}}$ is a symmetric matrix with eigenvalues equal to the square roots of the eigenvalues of $(\lambda_{2,\max} I_{\eta_1^*} - L_2)$, which are nonnegative.

Let $\underset{\sim}{\omega}_{2:3}$ be partitioned just like $\underset{\sim}{x}_{2:3}$, that is

$$\underset{\sim}{\omega}_{2:3} = \begin{bmatrix} \underset{\sim}{\omega}_2 \\ \underset{\sim}{\omega}_3^* \end{bmatrix} \tag{3.68}$$

where $\underset{\sim}{\omega}_2$ and $\underset{\sim}{\omega}_3^*$ are of orders s-1 and (r-1)(s-1)-p, respectively.

Theorem 3.2 represents the major result of this section.

Theorem 3.2

(a) $E(\underset{\sim}{\omega}_2) = \underset{\sim}{0}$; $E(\underset{\sim}{\omega}_3^*) = \underset{\sim}{0}$

(b) $\underset{\sim}{\omega}_2$ and $\underset{\sim}{\omega}_3^*$ are independent, normally distributed random vectors and have the following variance-covariance matrices:

$$\mathrm{Var}(\underset{\sim}{\omega}_2) = \Lambda_2^*\sigma_\beta^2 + \delta_2 I_{s-1}$$

$$\mathrm{Var}(\underset{\sim}{\omega}_3^*) = \delta_2 I_{(r-1)(s-1)-p} \tag{3.69}$$

where $\delta_2 = \sigma_{\alpha\beta}^2 + \lambda_{2,\max}\sigma_\epsilon^2$.

Proof:

The proof of Theorem 3.2 is very similar to the proof of Theorem 3.1 and is thus omitted. □

As a result of Theorem 3.2 we obtain

(i) $SS_2 = \underset{\sim}{\omega}_2'(\Lambda_2^*\sigma_\beta^2 + \delta_2 I_{s-1})^{-1}\underset{\sim}{\omega}_2 \sim \chi_{s-1}^2$ (3.70)

(ii) $SS_3^* = \underset{\sim}{\omega}_3^{*\prime}\underset{\sim}{\omega}_3^*/\delta_2 \sim \chi_{(r-1)(s-1)-p}^2$ (3.71)

(iii) SSS_2 and SS_3^* are independent.

A test statistic for testing $H_0:\sigma_\beta^2 = 0$ vs. $H_a:\sigma_\beta^2 \neq 0$ is then

$$F_2 = \frac{\underset{\sim}{\omega}_2'\underset{\sim}{\omega}_2/(s-1)}{\underset{\sim}{\omega}_3^{*\prime}\underset{\sim}{\omega}_3^*/\big((r-1)(s-1)-p\big)} \; . \tag{3.72}$$

Under H_0, $F_2 \sim F_{s-1,(r-1)(s-1)-p}$. Note that

$$E\left(\frac{\underset{\sim}{\omega}_2'\underset{\sim}{\omega}_2}{s-1}\right) = \bar{\theta}_2\sigma_\beta^2 + \delta_2 \tag{3.73}$$

where $\bar{\theta}_2$ is the average of the eigenvalues of $F_{2:3}^{11}$. Since $F_{2:3}$ (see

(3.60)) is p.d. it follows that $F_{2:3}^{-1}$ is p.d. Thus $F_{2:3}^{11}$ is p.d., implying that $\overline{\theta}_2 > 0$. Consequently we would reject H_0 at the γ-level of significance if $F_2 > F_{\gamma,s-1,(r-1)(s-1)-p}$.

A test for the variance component associated with the interaction effect is developed in the next section.

3.1.5 An Exact Test Concerning $\sigma_{\alpha\beta}^2$

In this section we develop an exact test for $\sigma_{\alpha\beta}^2$ by using $\underset{\sim}{v}_3$ (see (3.23)). We note at the onset that this test is equivalent to Thomsen's (1975) test.

From (3.24) - (3.26) we have

$$E(\underset{\sim}{v}_3) = \underset{\sim}{0}$$
$$\mathrm{Var}(\underset{\sim}{v}_3) = F_{33}\sigma_{\alpha\beta}^2 + G_{33}\sigma_\epsilon^2. \tag{3.74}$$

Since $\underset{\sim}{v}$ is a function of $\underset{\sim}{\bar{y}}$ (see (3.14) and (3.20)) and $\underset{\sim}{\bar{y}}$ is independent of SSE (Lemma B.6(v)), $\underset{\sim}{v}$ is also independent of SSE. Thus $\underset{\sim}{v}_3$ is independent of SSE as it is a subvector of $\underset{\sim}{v}$. Also, it clearly follows from (3.74) that

$$\underset{\sim}{v}_3'(F_{33}\sigma_{\alpha\beta}^2 + G_{33}\sigma_\epsilon^2)^{-1}\underset{\sim}{v}_3 \sim \chi^2_{(r-1)(s-1)-p}. \tag{3.75}$$

Because $SSE/\sigma_\epsilon^2 \sim \chi^2_{n-(rs-p)}$ a test statistic for testing $H_0:\sigma_{\alpha\beta}^2 = 0$ $H_a:\sigma_{\alpha\beta}^2 \neq 0$ is

$$F_3 = \frac{\underset{\sim}{v}_3' G_{33}^{-1} \underset{\sim}{v}_3 \Big/ \big((r-1)(s-1)-p\big)}{SSE \Big/ \big(n-(rs-p)\big)}. \tag{3.76}$$

Under H_0, $F_3 \sim F_{(r-1)(s-1)-p,n-(rs-p)}$. Note that

$$E\left[\frac{\underset{\sim}{v}_3' G_{33}^{-1} \underset{\sim}{v}_3}{(r-1)(s-1)-p}\right] = \bar{\theta}_3 \sigma_{\alpha\beta}^2 + \sigma_\epsilon^2 \tag{3.77}$$

where $\bar{\theta}_3$ is the average of the eigenvalues of $G_{33}^{-1}F_{33}$. Since both G_{33}^{-1} and F_{33} are p.d. $\bar{\theta}_3 > 0$ (see Graybill(1983, p. 397)). Hence we would reject H_0 for large values of F_3. In this case, we did not need to resample to construct the exact test.

3.1.6 A Numerical Example

An automobile manufacturer wished to study the effects of differences between drivers and differences between cars on gasoline consumption. Four drivers were selected at random; also five cars of the same model with manual transmission were randomly selected from the assembly line. Each driver drove each car a maximum of three times over a 40-mile test course and the miles per gallon were recorded. For technical reasons the drivers did not drive the cars the same number of times. Unfortunately, unforeseen mechanical problems forced the cancellation of some of the test drives. The data is recorded in Table 3.1

We first test for a significant interaction effect using (3.76). We find that

$F_3 = 0.7167$

numerator d.f. $= 9$

denominator d.f. $= 22$

p-value $= 0.6885$.

Thus there does not appear to be a significant interaction effect. The results concerning the main effects variance components are now presented.

For testing $H_0 : \sigma_\alpha^2 = 0$, regarding the driver variance component, we use (3.56). For this data we find that

$F_1 = 261.1971$

numerator d.f. $= 3$

denominator d.f. $= 9$

p-value < 0.0001

The test for the car variance component resulted in $\big($see (3.72)$\big)$

$F_2 = 71.2372$

numerator d.f. $= 4$

denominator d.f. $= 9$

p-value < 0.0001

All calculations were performed using S-plus.

Table 3.1 An example of an unbalanced random two-way model with missing cells

	B(car)				
A(driver)	1	2	3	4	5
1	25.3 25.2	28.1 28.9 30.0	24.8 25.1	28.4 27.9 26.7	
2	33.6 32.9 31.8	36.7 36.5	31.7 31.9	35.6 35.0	33.7 33.9 32.9
3	27.7 28.5	30.7 30.4		29.7 30.2 31.5	29.2 28.9
4	29.2 29.3	32.4 32.4		31.8 30.7	30.3 29.9

(source: Dr. André I. Khuri)

3.1.7 The Power of the Exact Tests

The approximate power of the exact test concerning σ_α^2 is derived in this section. Similar expressions can also be developed for the test for σ_β^2 and the test for $\sigma_{\alpha\beta}^2$. Our development is based on the results in Hirotsu (1979, pp. 578-579). The reader is referred to Appendix D.

Let Ψ_1 denote the power of the test for σ_α^2. Then from (3.56)

$$\Psi_1 = P\left[\frac{\underset{\sim}{\omega}_1'\underset{\sim}{\omega}_1/(r-1)}{\underset{\sim}{\omega}_3'\underset{\sim}{\omega}_3/[(r-1)(s-1)-p]} \geq F_{\gamma,r-1,(r-1)(s-1)-p} \,|\, H_a\right] \tag{3.78}$$

where $H_a : \sigma_\alpha^2 \neq 0$. Under H_a, $\underset{\sim}{\omega}_1'\underset{\sim}{\omega}_1$ is distributed like a random variable of the form (see Box (1954b))

$$\Upsilon = \sum_{j=1}^{r-1} (\lambda_j^* \sigma_\alpha^2 + \delta_1)\chi_j, \tag{3.79}$$

where the χ_j's are independent chi-squared variates with one degree of freedom, and λ_j^* is the j^{th} diagonal element of Λ_1^* $(j = 1, 2, \cdots, r-1)$; Λ_1^* is defined in (3.35) and δ_1 in (3.51). The exact distribution of Υ is complicated. To obtain approximate power values let

$$MS_3 = \frac{\underset{\sim}{\omega}_3'\underset{\sim}{\omega}_3}{(r-1)(s-1)-p} \tag{3.80}$$

$$H = \frac{\underset{\sim}{\omega}_1'\underset{\sim}{\omega}_1/cf}{MS_3/\delta_1} \tag{3.81}$$

$$h = \frac{\delta_1(r-1)}{cf} F_{\gamma, r-1, (r-1)(s-1)-p} \tag{3.82}$$

where in this case

$$c = \delta_1 \frac{\sum_{j=1}^{r-1} (\lambda_j^* \xi + 1)^2}{\sum_{j=1}^{r-1} (\lambda_j^* \xi + 1)} \tag{3.83}$$

$$f = \frac{\left[\sum_{j=1}^{r-1} (\lambda_j^* \xi + 1)\right]^2}{\sum_{j=1}^{r-1} (\lambda_j^* \xi + 1)^2} \tag{3.84}$$

$$\xi = \frac{\sigma_\alpha^2}{\delta_1} = \frac{\sigma_\alpha^2}{\sigma_{\alpha\beta}^2 + \lambda_{1,\max}\sigma_\epsilon^2} \tag{3.85}$$

Then

$$\begin{aligned}
\Psi_1 &= P(H \geq h \mid H_a) \doteq P(F_{f,f_2} \geq h) \\
&+ \left[\rho / \left\{3(f+2)(f+4)B(\tfrac{1}{2}f, \tfrac{1}{2}f_2)\right\}\right] \\
&\times \left(1 + \frac{fh}{f_2}\right)^{-\frac{1}{2}(f+f_2)} \left(\frac{fh}{f_2}\right)^{\frac{1}{2}f} \\
&\times \left[(f+2)(f+4) - \frac{2(f+f_2)(f+4)}{1 + f_2/(fh)} + \frac{(f+f_2+2)(f+f_2)}{\{1+f_2/(fh)\}^2}\right]
\end{aligned} \tag{3.86}$$

where

$$f_2 = (r-1)(s-1)-p \tag{3.87}$$

$$\rho = \frac{\left[\sum_{j=1}^{r-1}(\lambda_j^*\xi+1)\right]\left[\sum_{j=1}^{r-1}(\lambda_j^*\xi+1)^3\right]}{\left[\sum_{j=1}^{r-1}(\lambda_j^*\xi+1)^2\right]^2} - 1 \tag{3.88}$$

and $B(m_1,m_2)$ is the beta function. We note that from (3.86) the power of the exact test depends on the design parameters $\lambda_{1,\max}, \lambda_1^*, \cdots, \lambda_{r-1}^*$; the level of significance; and the ratios $\sigma_\alpha^2/\sigma_\epsilon^2$ and $\sigma_{\alpha\beta}^2/\sigma_\epsilon^2$ of the variance components.

3.1.8 Concluding Remarks

The key to the construction of the exact tests for the main effects' variance components were the random vectors $\underset{\sim}{\omega}_{1:3}$ and $\underset{\sim}{\omega}_{2:3}$ in (3.49) and (3.67), respectively. Both of these vectors represent a linear combination between a function of $\underset{\sim}{y}$ and a portion of the error vector.

We note that C_1 in (3.49) and C_1^* in (3.67) are not unique. Thus the values of the test statistics F_1 and F_2 (see (3.56) and (3.72), respectively) will depend on the choice for C_1 and C_1^*, respectively. However, the distributions of $\underset{\sim}{\omega}_{1:3}$ and $\underset{\sim}{\omega}_{2:3}$ are invariant to these choices. It is in this sense that our methodology is similar to techniques involving resampling.

We also presented an exact test for the variance component associated with the interaction effect. This test is similar to Thomsen's (1975) procedure. Seely and El-Bassiouni (1983) point out that Thomsen's (1975) test for $\sigma^2_{\alpha\beta}$ is a Wald's test. Since our test is based on the same degrees of freedom as Thomsen's test, we conclude that our test for $\sigma^2_{\alpha\beta}$ is also a Wald's test. Consequently, F_3 in (3.76) can be equivalently expressed as

$$F_3 = \frac{R\big((\underset{\sim}{\alpha\beta}) \mid \mu, \underset{\sim}{\alpha}, \underset{\sim}{\beta}\big)/[(r-1)(s-1)-p]}{SSE/\big(n-(rs-p)\big)} \tag{3.89}$$

where $R\big((\underset{\sim}{\alpha\beta}) \mid \mu, \underset{\sim}{\alpha}, \underset{\sim}{\beta}\big)$ represents the sum of squares for $(\underset{\sim}{\alpha\beta})$ adjusted for μ, $\underset{\sim}{\alpha}$, and $\underset{\sim}{\beta}$. Here, $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$ are treated as if they were fixed effects $\big($see Seely and El-Bassiouni (1983, p. 198)$\big)$.

Consider again $\underset{\sim}{\omega}_{1:3}$ defined in (3.49). It is easy to verify that $\underset{\sim}{\omega}_{1:3}$ can be written as

$$\underset{\sim}{\omega}_{1:3} = \Lambda_1^{\frac{1}{2}}\underset{\sim}{\alpha} + \underset{\sim}{\varepsilon}_1^* \tag{3.90}$$

where

$$\Lambda_1^{\frac{1}{2}} = \begin{pmatrix} \Lambda_1^{*\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix} \tag{3.91}$$

$\big($see (3.35)$\big)$, and

$$\underset{\sim}{\varepsilon}_1^* \sim N(\underset{\sim}{0},\ \delta_1 I_{s(r-1)-p}). \tag{3.92}$$

δ_1 is defined in (3.51). Now, we have

$$\operatorname{rank}(\Lambda_1^{\frac{1}{2}}) = r-1$$

and (using similar notation to that of Seely and El-Bassiouni),

$$f_W = s(r-1)-p-(r-1) = (r-1)(s-1)-p$$

$$k_W = r-1.$$

Thus, the test statistic F_1 is a Wald's test based on the $\underset{\sim}{\omega}_{1:3}$ model. Likewise, we can also show that F_2 is a Wald's test based on the $\underset{\sim}{\omega}_{2:3}$ model.

3.2 The Unbalanced 3-Fold Nested Random Model

3.2.1 Introduction

The unbalanced 3-fold nested random model can be expressed as

$$Y_{ijk\ell} = \mu + \alpha_i + \beta_{ij} + \gamma_{ijk} + \epsilon_{ijk\ell} \tag{3.93}$$

($i = 1,2,\cdots,a$; $j = 1,2,\cdots,b_i$; $k = 1,2,\cdots,c_{ij}$; $\ell = 1,2,\cdots,n_{ijk}$), where μ is an unknown constant parameter; α_i, β_{ij}, γ_{ijk} are random-effects associated with the first-stage, second-stage and third-stage nested factors, respectively; and $\epsilon_{ijk\ell}$ is the random error component. We assume that α_i, β_{ij}, γ_{ijk}, and $\epsilon_{ijk\ell}$ are independently distributed normal random variables with zero means and variances σ_α^2, σ_β^2, σ_γ^2, and σ_ϵ^2, respectively. We assume there are no missing cells. If n is the total number of observations and c and b the total number of levels of the third- and second-stage nested factors, respectively, then

$$n = \sum_{i,j,k} n_{ijk}$$

$$c = \sum_{i,j} c_{ij} \qquad (3.94)$$

$$b = \sum_{i} b_{i} .$$

We are concerned with constructing exact tests for σ_α^2 and σ_β^2. We again need to use resampling to achieve our goal. The results given in Seely and El-Bassiouni (1983) will allow us to conclude that our tests are Wald's tests. We note that an exact test for σ_γ^2 is known to exist based on comparing $R(\gamma|\mu,\alpha,\beta)$ to SSE where $R(\gamma|\mu,\alpha,\beta)$ is the sum of squares for γ adjusted for μ,α, and β (treating α,β and γ as fixed effects); and SSE is the error sum of squares. That is, to test $H_0:\sigma_\gamma^2=0$ vs. $H_a:\sigma_\gamma^2 \neq 0$ we would reject H_0 for large values of F_γ where

$$F_\gamma = \frac{R(\gamma|\mu,\alpha,\beta)/(c-b)}{SSE/(n-c)} .$$

Under H_0, $F_\gamma \sim F_{c-b,n-c}$ (see, e.g., Searle (1971)). Resampling was not needed to develop this test.

Now, in matrix notation, model (3.93) becomes

$$\underset{\sim}{y} = \mu \underset{\sim}{1}_n + X_1\underset{\sim}{\alpha} + X_2\underset{\sim}{\beta} + X_3\underset{\sim}{\gamma} + \underset{\sim}{\epsilon} \qquad (3.95)$$

where $\underset{\sim}{y}$ is the $n \times 1$ vector of observations; $\underset{\sim}{1}_n$ is a vector of ones of dimension n; X_1, X_2, and X_3 are known matrices of one's and zero's (see (3.97) - (3.99)); $\underset{\sim}{\alpha}$, $\underset{\sim}{\beta}$, and $\underset{\sim}{\gamma}$ are random vectors whose elements are the α_i, β_{ij}, and γ_{ijk}, respectively; and $\underset{\sim}{\epsilon}$ is the $n \times 1$ vector of errors. If n_i and n_{ij} are defined as

$$n_i = \sum_{j,k} n_{ijk} \qquad (i = 1, 2, \cdots, a)$$

$$n_{ij} = \sum_{k} n_{ijk} \quad (i = 1, 2, \cdots, a;\ j = 1, 2, \cdots, b_i) \tag{3.96}$$

then,

$$X_1 = \overset{a}{\underset{i=1}{\oplus}} \underset{\sim}{1}_{n_i} \tag{3.97}$$

$$X_2 = \overset{a}{\underset{i=1}{\oplus}} \overset{b_i}{\underset{j=1}{\oplus}} \underset{\sim}{1}_{n_{ij}} \tag{3.98}$$

$$X_3 = \overset{a}{\underset{i=1}{\oplus}} \overset{b_i}{\underset{j=1}{\oplus}} \overset{c_{ij}}{\underset{k=1}{\oplus}} \underset{\sim}{1}_{n_{ijk}} \tag{3.99}$$

From (3.95) we have that

$$\underset{\sim}{y} \sim N(\mu \underset{\sim}{1}_n,\ \Omega)$$

where

$$\Omega = (\underset{i}{\oplus} J_{n_i})\sigma_\alpha^2 + (\underset{i,j}{\oplus} J_{n_{ij}})\sigma_\beta^2 + (\underset{i,j,k}{\oplus} J_{n_{ijk}})\sigma_\gamma^2 + I_n \sigma_\epsilon^2. \tag{3.100}$$

Now, define $\bar{y}_{ijk}$ as

$$\bar{y}_{ijk} = \frac{1}{n_{ijk}} \sum_{\ell=1}^{n_{ijk}} y_{ijk\ell} \tag{3.101}$$

$(i = 1,2,\cdots,a;\ j = 1,2,\cdots,b_i;\ k = 1,2,\cdots,c_{ij})$. Then from (3.93) we have

$$\bar{y}_{ijk} = \mu + \alpha_i + \beta_{ij} + \gamma_{ijk} + \bar{\epsilon}_{ijk} \tag{3.102}$$

$(i = 1,2,\cdots,a;\ j = 1,2,\cdots,b_i;\ k = 1,2,\cdots,c_{ij})$, where

$$\bar{\epsilon}_{ijk} = \frac{1}{n_{ijk}} \sum_{\ell=1}^{n_{ijk}} \epsilon_{ijk\ell}.$$

In matrix form, (3.102) becomes

$$\underset{\sim}{\bar{y}} = \mu \underset{\sim}{1}_c + A_1 \underset{\sim}{\alpha} + A_2 \underset{\sim}{\beta} + I_c \underset{\sim}{\gamma} + \underset{\sim}{\bar{\epsilon}} \tag{3.103}$$

where $\underset{\sim}{\bar{y}}$ is the $c\times 1$ vector of means; $\underset{\sim}{\bar{\epsilon}}$ is the $c\times 1$ vector $\underset{\sim}{\bar{\epsilon}} = (\bar{\epsilon}_{111}, \bar{\epsilon}_{112}, \cdots, \bar{\epsilon}_{ab_a c_{ab_a}})'$; and A_1 and A_2 are defined as

$$A_1 = \bigoplus_{i=1}^{a} \underset{\sim}{1}_{c_i} \tag{3.104}$$

$$A_2 = \bigoplus_{i=1}^{a} \bigoplus_{j=1}^{b_i} \underset{\sim}{1}_{c_{ij}} \tag{3.105}$$

where

$$c_i = \sum_j c_{ij} \quad (i = 1, 2, \cdots, a). \tag{3.106}$$

We note that

$$\text{range}(\underset{\sim}{1}_c) \subset \text{range}(A_1) \subset \text{range}(A_2).$$

In fact,

$$\begin{aligned} \underset{\sim}{1}_c &= A_1 \underset{\sim}{1}_a = A_2 \underset{\sim}{1}_b \\ A_1 &= A_2 \left(\overset{a}{\underset{i=1}{\oplus}} \underset{\sim}{1}_{b_i} \right). \end{aligned} \tag{3.107}$$

From (3.103) we have

$$\underset{\sim}{\bar{y}} \sim N(\mu \underset{\sim}{1}_c, \ \Sigma)$$

where

$$\Sigma = \tilde{A}_1 \sigma_\alpha^2 + \tilde{A}_2 \sigma_\beta^2 + I_c \sigma_\gamma^2 + K \sigma_\epsilon^2 \tag{3.108}$$

and

$$\tilde{A}_1 = A_1 A_1' = \overset{a}{\underset{i=1}{\oplus}} J_{c_i} \tag{3.109}$$

$$\tilde{A}_2 = A_2A_2' = \bigoplus_{i=1}^{a} \bigoplus_{j=1}^{b_i} J_{c_{ij}} \tag{3.110}$$

$$K = \bigoplus_{i=1}^{a} \bigoplus_{j=1}^{b_i} \bigoplus_{k=1}^{c_{ij}} n_{ijk}^{-1}. \tag{3.111}$$

Besides the assumptions concerning the random effects, we will also assume:

(i) The model used is model (3.93) (or model (3.95))

(ii) $b-a>0$ (b is defined in (3.94)) (3.112)

(iii) $n>2c-1$ (n and c are defined in (3.94))

(iv) $c>2b-1$.

The last two assumptions (along with the others) are needed to insure the validity of our procedures. Neither is overly restrictive. Assumption (iii) can be satisfied if, for example, $n_{ijk}\geq 2$ for all (i,j,k) while Assumption (iv) will hold if, for example, $c_{ij}\geq 2$ for all (i,j).

3.2.2. Preliminary Development

In this section we use the last $c-1$ rows of the $c\times c$ Helmert matrix (see Searle (1982), p. 71) to transform model (3.103) into one that has a zero mean vector. We also present some results concerning the ranks of the model matrices in the transformed model.

Let H_1 be the $(c-1)\times c$ matrix resulting from the deletion of the first row of the $c\times c$ Helmert matrix. H_1 has orthonormal rows and is such that $H_1\underset{\sim}{1}_c=\underset{\sim}{0}_{c-1}$. Define the $(c-1)\times 1$ vector $\underset{\sim}{u}$ as

$$\underset{\sim}{u}=H_1\underset{\sim}{\bar{y}}. \tag{3.113}$$

Then,

$$E(\underset{\sim}{u})=H_1E(\underset{\sim}{\bar{y}})=\mu H_1\underset{\sim}{1}_c=\underset{\sim}{0}_{c-1}$$

$$\mathrm{Var}(\underset{\sim}{u})=H_1\tilde{A}_1H_1'\sigma_\alpha^2+H_1\tilde{A}_2H_1'\sigma_\beta^2+I_{c-1}\sigma_\gamma^2+L\sigma_\epsilon^2 \tag{3.114}$$

where

$$L=H_1KH_1'. \tag{3.115}$$

Recalling (3.104) and (3.105), it is easily seen that

$$\begin{aligned}\mathrm{rank}(A_1)&=a\\ \mathrm{rank}(A_2)&=b.\end{aligned} \tag{3.116}$$

The next lemma provides information on the ranks of $H_1\tilde{A}_1H_1'$ and $H_1\tilde{A}_2H_1'$.

Lemma 3.3

(a) $\text{rank}(H_1\tilde{A}_1H_1') = a-1$

(b) $\text{rank}(H_1\tilde{A}_2H_1') = b-1$

(c) $\text{rank}(H_1\tilde{A}_1H_1' + H_1\tilde{A}_2H_1') = b-1$.

Proof:

(a) $$\begin{aligned}\text{rank}(H_1\tilde{A}_1H_1') &= \text{rank}(H_1A_1A_1'H_1') \\ &= \text{rank}(H_1A_1) \\ &\leq \text{rank}(A_1) = a. \qquad \left(\text{see } (3.116)\right)\end{aligned}$$

However, from (3.107), $\underset{\sim}{1}_c = A_1\underset{\sim}{1}_a$. Since $H_1\underset{\sim}{1}_c = \underset{\sim}{0}_{c-1}$ it follows that the a columns of H_1A_1 are linearly dependent. Thus $\text{rank}(H_1A_1) \leq a-1 \Rightarrow \text{rank}(H_1\tilde{A}_1H_1') \leq a-1$. Conversely, by the Frobenius Inequality (see Lemma A.1), we have

$$\begin{aligned}\text{rank}(H_1\tilde{A}_1H_1') &= \text{rank}(H_1A_1) \\ &\geq \text{rank}(H_1) + \text{rank}(A_1) - \text{rank}(I_c)\end{aligned}$$

$$= c - 1 + a - c = a - 1.$$

Thus, $\operatorname{rank}(H_1\tilde{A}_1H_1') = a - 1$.

(b) Similar to (a).

(c) Since $\operatorname{range}(A_1) \subset \operatorname{range}(A_2)$ we have that $\operatorname{range}(H_1A_1) \subset \operatorname{range}(H_1A_2)$. Consequently,

$$\operatorname{range}\ (H_1A_1A_1'H_1') = \operatorname{range}(H_1A_1) \subset \operatorname{range}(H_1A_2) = \operatorname{range}(H_1A_2A_2'H_1').$$

Thus there exists a matrix B such that

$$H_1\tilde{A}_1H_1' = (H_1\tilde{A}_2H_1')B.$$

(Since both $H_1\tilde{A}_1H_1'$ and $H_1\tilde{A}_2H_1'$ are n.n.d. B must be n.n.d.) Hence,

$$\begin{aligned} \operatorname{rank}(H_1\tilde{A}_1H_1' + H_1\tilde{A}_2H_1') &= \operatorname{rank}\left[(H_1\tilde{A}_2H_1')B + H_1\tilde{A}_2H_1'\right] \\ &= \operatorname{rank}\left[H_1\tilde{A}_2H_1'(B+I)\right] \\ &= \operatorname{rank}(H_1\tilde{A}_2H_1') \qquad (B+I \text{ has full rank}) \\ &= b - 1 \qquad \left(\text{see (b)}\right). \end{aligned}$$

□

3.2.3 An Exact Test Concerning σ_β^2

Introduction. In this section we construct an exact test for the variance component associated with the second-stage nested factor. The idea is to first make a transformation of model (3.113) based on resampling from the error vector. This transformation will result in a model that is a special case of the model described in Seely and El-Bassiouni (1983, p. 198). We will then apply the results in Seely and El-Bassiouni's Equation 3.2 to develop the exact procedure.

Utilizing resampling. Recall model (3.95). Let W be the partitioned matrix

$$W = [\underset{\sim}{1}_n : X_1 : X_2 : X_3] \tag{3.117}$$

and let R be the matrix

$$R = \underset{\sim}{I} - W(W'W)^{-}W'. \tag{3.118}$$

Since

$$\text{range}(\underset{\sim}{1}_n) \subset \text{range}(X_1) \subset \text{range}(X_2) \subset \text{range}(X_3),$$

(3.118) can be expressed as

$$R = I_n - X_3(X_3'X_3)^{-1}X_3'. \tag{3.119}$$

The error sum of squares associated with model (3.95) is given by

$$\mathrm{SSE} = \underset{\sim}{y}'\mathrm{R}\underset{\sim}{y}. \tag{3.120}$$

It is known that R is symmetric, idempotent, and has rank n-c. Furthermore, $\mathrm{SSE}/\sigma_\epsilon^2$ has the chi-squared distribution with n-c degrees of freedom independently of $\bar{\underset{\sim}{y}}$ (see, e.g., Lemma B.6). We can express R as

$$\mathrm{R} = \mathrm{C}\Gamma\mathrm{C}' \tag{3.121}$$

where C is an orthogonal matrix and Γ is a diagonal matrix whose first n-c diagonal elements are equal to unity and the remaining c entries are equal to zero. By assumption (3.112)(iii), we can partition C and Γ as

$$\begin{aligned} \mathrm{C} &= [\mathrm{C}_1 : \mathrm{C}_2 : \mathrm{C}_3] \\ \Gamma &= \mathrm{diag}(\mathrm{I}_{\eta_1}, \mathrm{I}_{\eta_2}, 0) \end{aligned} \tag{3.122}$$

where

$$\begin{aligned} \eta_1 &= c-1 \\ \eta_2 &= n-2c+1 > 0 \end{aligned} \tag{3.123}$$

and C_1, C_2, and C_3 are matrices of orders $n \times \eta_1$, $n \times \eta_2$, and $n \times c$, respectively. Note that $\eta_1 + \eta_2 = n - c = \text{rank}(R)$.

From these partitionings and the fact that (recall C is orthogonal)

$$\begin{aligned} C_i' C_i &= I \qquad (i = 1,2,3) \\ C_i' C_j &= 0 \qquad (i \neq j), \end{aligned} \tag{3.124}$$

we obtain

$$R = C_1 C_1' + C_2 C_2'. \tag{3.125}$$

Now, define the (c-1)×1 random vector $\underset{\sim}{\omega}$ as

$$\underset{\sim}{\omega} = \underset{\sim}{u} + (\lambda_{\max} I_{\eta_1} - L)^{\frac{1}{2}} C_1' \underset{\sim}{y} \tag{3.126}$$

where $\lambda_{\max}$ is the largest eigenvalue of L $\big($see (3.115)$\big)$ and C_1' is the $n \times \eta_1$ matrix defined in (3.122). The matrix $(\lambda_{\max} I_{\eta_1} - L)$ is p.s.d. Hence $(\lambda_{\max} I_{\eta_1} - L)^{\frac{1}{2}}$ is well-defined with eigenvalues equal to the square roots of the eigenvalues of $(\lambda_{\max} I_{\eta_1} - L)$, which are nonnegative. The first major result of this section is given in the next theorem which provides some distributional properties for $\underset{\sim}{\omega}$.

Theorem 3.3

The random vector $\underset{\sim}{\omega}$ is normally distributed with

$$E(\underset{\sim}{\omega}) = \underset{\sim}{0}_{c-1}$$

$$\mathrm{Var}(\underset{\sim}{\omega}) = H_1\tilde{A}_1H_1'\sigma_\alpha^2 + H_1\tilde{A}_2H_1'\sigma_\beta^2 + \delta I_{c-1} \tag{3.127}$$

where $\delta = \sigma_\gamma^2 + \lambda_{\max}\sigma_\epsilon^2$.

Proof:

The fact that $\underset{\sim}{\omega}$ is normally distributed is clear. We know that $E(\underset{\sim}{u}) = \underset{\sim}{0}$ $\Big($see (3.114)$\Big)$. From Lemma B.6 (ii) we have that $R\underset{\sim}{1}_n = \underset{\sim}{0}$. Since $\mathrm{range}(C_1) \subset \mathrm{range}(R)$, it follows that $C_1'\underset{\sim}{1}_n = \underset{\sim}{0}$ (recall that R is symmetric). Hence

$$\begin{aligned} E(\underset{\sim}{\omega}) &= E(\underset{\sim}{u}) + (\lambda_{\max}I_{\eta_1} - L)^{\frac{1}{2}}C_1'E(\underset{\sim}{y}) \\ &= \underset{\sim}{0} + (\lambda_{\max}I_{\eta_1} - L)^{\frac{1}{2}}C_1'(\underset{\sim}{1}_n)\mu \\ &= \underset{\sim}{0}. \end{aligned}$$

Now, we claim that $\underset{\sim}{u}$ and $C_1'\underset{\sim}{y}$ are independent in the expression given for $\underset{\sim}{\omega}$ in (3.126). To verify this recall that SSE is independent of $\underset{\sim}{\bar{y}}$

(see Lemma (B.6)(v)). Consequently,

$$D\Omega R = 0$$

where D is such that $D\underset{\sim}{y} = \underset{\sim}{\bar{y}}$, and Ω is defined in (3.100). Since range(C_1) $\subset$ range(R) it follows that

$$D\Omega C_1 = 0.$$

Thus $\underset{\sim}{\bar{y}}$ is independent of $C_1'\underset{\sim}{y}$. Since $\underset{\sim}{u}$ is a function of $\underset{\sim}{\bar{y}}$, $\underset{\sim}{u}$ is also independent of $C_1'\underset{\sim}{y}$ as claimed. Hence,

$$\mathrm{Var}(\underset{\sim}{\omega}) = \mathrm{Var}(\underset{\sim}{u}) + (\lambda_{\max} I_{\eta_1} - L)^{\frac{1}{2}} C_1' \Omega C_1 (\lambda_{\max} I_{\eta_1} - L)^{\frac{1}{2}}. \qquad (3.128)$$

From Lemma B.6(ii), $R\Omega R = R^2\sigma_\epsilon^2$. Thus since range($C_1$) $\subset$ range(R) and R is idempotent we have

$$C_1'\Omega C_1 = C_1'C_1\sigma_\epsilon^2 = \sigma_\epsilon^2 I_{\eta_1}.$$

The last equality follows from (3.124). Therefore we can express (3.128) as

$$\mathrm{Var}(\underset{\sim}{\omega}) = \mathrm{Var}(\underset{\sim}{u}) + (\lambda_{\max} I_{\eta_1} - L)\sigma_\epsilon^2$$

$$= H_1\tilde{A}_1H_1'\sigma_\alpha^2 + H_1\tilde{A}_2H_1'\sigma_\beta^2 + I_{c-1}\sigma_\gamma^2 + L\sigma_\epsilon^2 + (\lambda_{\max} I_{\eta_1} - L)\sigma_\epsilon^2$$

$$= H_1\tilde{A}_1H_1'\sigma_\alpha^2 + H_1\tilde{A}_2H_1'\sigma_\beta^2 + \delta I_{c-1}$$

where $\delta = \sigma_\gamma^2 + \lambda_{\max}\sigma_\epsilon^2$. □

From Theorem 3.3 we note that model (3.126) can also be written as

$$\underset{\sim}{\omega} = B_1\underset{\sim}{\alpha} + B_2\underset{\sim}{\beta} + \underset{\sim}{\epsilon}^* \tag{3.129}$$

where $B_1 = H_1A_1$; $B_2 = H_1A_2$; and $\underset{\sim}{\epsilon}^* = H_1\underset{\sim}{\gamma} + H_1\underset{\sim}{\bar{\epsilon}} + (\lambda_{\max}I_{\eta_1} - L)^{\frac{1}{2}}C_1'\underset{\sim}{\epsilon} \sim N(\underset{\sim}{0}, \delta I_{c-1})$ independently of $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$. This model in the form required by Seely and El-Bassiouni (1983, p. 198). We are now in position to construct an exact test for testing $H_0{:}\sigma_\beta^2 = 0$ vs. $H_a{:}\sigma_\beta^2 \neq 0$.

The exact test. Let us start with some notation. $SSE_{\underset{\sim}{\omega}}$ will denote the error sum of squares associated with model (3.129). That is,

$$SSE_{\underset{\sim}{\omega}} = \underset{\sim}{\omega}'\Big(I_{c-1} - [B_1{:}B_2]([B_1{:}B_2]'[B_1{:}B_2])^-[B_1{:}B_2]'\Big)\underset{\sim}{\omega}. \tag{3.130}$$

Also, let $R_{\underset{\sim}{\omega}}(\underset{\sim}{\beta}|\underset{\sim}{\alpha})$ denote the sum of squares for $\underset{\sim}{\beta}$ adjusted for $\underset{\sim}{\alpha}$ (treating $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$ as fixed) in model (3.129). Thus

$$R_{\underset{\sim}{\omega}}(\underset{\sim}{\beta}|\underset{\sim}{\alpha}) = \underset{\sim}{\omega}'\Big(I_{c-1} - B_1(B_1'B_1)^-B_1'\Big)\underset{\sim}{\omega} - SSE_{\underset{\sim}{\omega}}. \tag{3.131}$$

Since $\mathrm{range}(B_1) \subset \mathrm{range}(B_2)$, (3.130) and (3.131) can be equivalently expressed as

$$\mathrm{SSE}_{\underset{\sim}{\omega}} = \underset{\sim}{\omega}'\Big(\mathrm{I}_{c-1} - \mathrm{B}_2(\mathrm{B}_2'\mathrm{B}_2)^{-}\mathrm{B}_2'\Big)\underset{\sim}{\omega} \qquad (3.132)$$

$$\mathrm{R}_{\underset{\sim}{\omega}}(\underset{\sim}{\beta}\,|\,\underset{\sim}{\alpha}) = \underset{\sim}{\omega}'\Big(\mathrm{B}_2(\mathrm{B}_2'\mathrm{B}_2)^{-}\mathrm{B}_2' - \mathrm{B}_1(\mathrm{B}_1'\mathrm{B}_1)^{-}\mathrm{B}_1'\Big)\underset{\sim}{\omega}. \qquad (3.133)$$

The second and last major result of this section is given in the next theorem.

Theorem 3.4

a) A g-inverse of $\mathrm{B}_j'\mathrm{B}_j = \mathrm{A}_j'\mathrm{H}_1'\mathrm{H}_1\mathrm{A}_j$ is given by $(\mathrm{A}_j'\mathrm{H}_1'\mathrm{H}_1\mathrm{A}_j)^{-} = (\mathrm{A}_j'\mathrm{A}_j)^{-1}$ $(j=1,2)$.

b) $\mathrm{R}_{\underset{\sim}{\omega}}(\underset{\sim}{\beta}\,|\,\underset{\sim}{\alpha})/\delta \sim \chi^2_{b-a}$, under $\mathrm{H}_0{:}\sigma^2_\beta = 0$

c) $\mathrm{SSE}_{\underset{\sim}{\omega}}/\delta \sim \chi^2_{c-b}$

d) $\mathrm{R}_{\underset{\sim}{\omega}}(\underset{\sim}{\beta}\,|\,\underset{\sim}{\alpha})$ is independent of $\mathrm{SSE}_{\underset{\sim}{\omega}}$

Proof:

a) Since $\mathrm{H}_1\mathrm{H}_1' = \mathrm{I}_{c-1}$ and $\mathrm{H}_1\underset{\sim}{1}_c = \underset{\sim}{0}_{c-1}$. It follows that $\mathrm{H}_1'\mathrm{H}_1 = \mathrm{I}_c - \frac{1}{c}\mathrm{J}_c$ where $\mathrm{J}_c = \underset{\sim}{1}_c\underset{\sim}{1}_c'$. Also recall that $\underset{\sim}{1}_c = \mathrm{A}_1\underset{\sim}{1}_a = \mathrm{A}_2\underset{\sim}{1}_b$ $\Big($see (3.107)$\Big)$. Thus

$$\mathrm{A}_j'\mathrm{H}_1'\mathrm{H}_1\mathrm{A}_j = \mathrm{A}_j'(\mathrm{I}_c - \tfrac{1}{c}\mathrm{J}_c)\mathrm{A}_j$$

$$= A'_jA_j - \tfrac{1}{c}A'_jJ_cA_j$$

$$= A'_jA_j - \tfrac{1}{c}A'_jA_jJ_tA'_jA_j \qquad (j=1,2),$$

where

$$t = \begin{cases} a & \text{if } j=1 \\ b & \text{if } j=2. \end{cases} \tag{3.134}$$

Therefore,

$$A'_jH'_1H_1A_j(A'_jA_j)^{-1}A'_jH'_1H_1A_j$$

$$= (A'_jA_j - \tfrac{1}{c}A'_jA_jJ_tA'_jA_j)(A'_jA_j)^{-1}(A'_jA_j - \tfrac{1}{c}A'_jA_jJ_tA'_jA_j)$$

$$= (I_t - \tfrac{1}{c}A'_jA_jJ_t)(A'_jA_j - \tfrac{1}{c}A'_jA_jJ_tA'_jA_j)$$

$$= A'_jA_j - \tfrac{2}{c}A'_jA_jJ_tA'_jA_j + \tfrac{1}{c^2}A'_jA_jJ_tA'_jA_jJ_tA'_jA_j$$

$$= A'_jA_j - \tfrac{2}{c}A'_jJ_cA_j + \tfrac{1}{c^2}A'_jJ_cA_jJ_tA'_jA_j$$

$$= A'_jA_j - \tfrac{2}{c}A'_jJ_cA_j + \tfrac{1}{c^2}A'_jJ_cJ_cA_j$$

$$= A'_jA_j - \tfrac{1}{c}A'_jJ_cA_j$$

$$= A_j(I_c - \tfrac{1}{c}J_c)A_j = A'_jH'_1H_1A_j.$$

Although (b), (c), and (d) follow from Seely and El-Bassiouni's (1983) Equation (3.2), we nevertheless prove these results for the sake of completeness.

b) First note that (3.133) can be expressed as

$$\underset{\sim}{\omega}'(P_2 - P_1)\underset{\sim}{\omega} \tag{3.135}$$

where

$$P_j = H_1A_j(A_j'A_j)^{-1}A_j'H_1' \quad (j = 1,2). \tag{3.136}$$

We need to verify that $(P_2 - P_1)\big(\mathrm{Var}(\underset{\sim}{\omega})\big)/\delta$ is idempotent of rank b-a when $\sigma_\beta^2 = 0$ $\big($see Lemma (B.4)$\big)$. Recall from Theorem 3.3 that

$$\mathrm{Var}(\underset{\sim}{\omega}) = H_1A_1A_1'H_1'\sigma_\alpha^2 + H_1A_2A_2'H_1'\sigma_\beta^2 + \delta I_{c-1}.$$

Since P_j is idempotent of rank t-1 $\big($see part (a) of this theorem, Lemma (3.3), and (3.134)$\big)$ and range$(P_1) \subset$ range(P_2) it follows that there exists a matrix $\mathcal{O}$ such that

$$P_1 = P_2\mathcal{O}$$

$$= \mathcal{O}'P_2. \quad \text{(by symmetry)}$$

Thus

$$P_1P_2 = \mathcal{O}'P_2P_2 = \mathcal{O}'P_2 = P_1 .$$

Consequently, $P_2 - P_1$ is idempotent and has rank $(b\text{-}1) - (a\text{-}1) = b\text{-}a$. Now, since $\text{range}(H_1A_1A_1'H_1') \subset \text{range}(H_1A_2A_2'H_1') = \text{range}(P_2)$, we have

$$\begin{aligned}
(P_2 - P_1)\Big(\text{Var}(\underset{\sim}{\omega})\Big) &= (I - P_1)P_2\text{Var}(\underset{\sim}{\omega}) \qquad (P_1P_2 = P_1) \\
&= (I - P_1)[H_1A_1A_1'H_1'\sigma_\alpha^2 + H_1A_2A_2'H_1'\sigma_\beta^2 + \delta P_2] \\
&= (I - P_1)H_1A_2A_2'H_1'\sigma_\beta^2 + \delta(I - P_1)P_2 \qquad (3.137) \\
&= \delta(P_2 - P_1) \qquad (\text{when } \sigma_\beta^2 = 0).
\end{aligned}$$

Formula (3.137) follows since $\text{range}(H_1A_1A_1'H_1') = \text{range}(P_1)$ and $(I - P_1)P_1 = 0$. Consequently, $(P_2 - P_1)\Big(\text{var}(\underset{\sim}{\omega})\Big)/\delta$ is idempotent of rank b-a. Thus from Lemma B.4 we have

$$R_{\underset{\sim}{\omega}}(\underset{\sim}{\beta}\,|\,\underset{\sim}{\alpha})/\delta \sim \chi^2_{b-a} \qquad \text{under } H_0\text{:}\sigma_\beta^2 = 0.$$

c) $\text{SSE}_{\underset{\sim}{\omega}}$ in (3.132) can be expressed as

$$\text{SSE}_{\underset{\sim}{\omega}} = \underset{\sim}{\omega}'(I_{c-1} - P_2)\underset{\sim}{\omega}.$$

In the expression for $\mathrm{Var}(\underset{\sim}{\omega})$, $\mathrm{range}(H_1A_1A_1'H_1) \subset \mathrm{range}(H_1A_2A_2'H_1')$ $= \mathrm{range}(P_2)$. Thus since $(I-P_2)P_2 = 0$ it follows that $(I-P_2)\big(\mathrm{Var}(\underset{\sim}{\omega})\big)/\delta = I - P_2$ and therefore has rank $(c\text{-}1)-(b\text{-}1) = c\text{-}b$. Thus from Lemma B.4,

$$\mathrm{SSE}_{\underset{\sim}{\omega}}/\delta \sim \chi^2_{c\text{-}b}.$$

d) To show that $R_{\underset{\sim}{\omega}}(\underset{\sim}{\beta}\,|\,\underset{\sim}{\alpha})$ is independent of $\mathrm{SSE}_{\underset{\sim}{\omega}}$ we need to verify that

$$(P_2-P_1)\big(\mathrm{Var}(\underset{\sim}{\omega})\big)(I_{c\text{-}1}-P_2) = 0$$

(see Lemma B.2). From (c) we have

$$\big(\mathrm{Var}(\underset{\sim}{\omega})\big)(I-P_2) = \delta(I-P_2).$$

Thus

$$(P_2-P_1)\big(\mathrm{Var}(\underset{\sim}{\omega})\big) = \delta(P_2-P_1)(I-P_2) = 0.$$

The last equality follows since $\mathrm{range}(P_1) \subset \mathrm{range}(P_2)$. Consequently $R_{\underset{\sim}{\omega}}(\underset{\sim}{\beta}\,|\,\underset{\sim}{\alpha})$ and $\mathrm{SSE}_{\underset{\sim}{\omega}}$ are independent. □

We can conclude from Theorem 3.4 that

$$F_\beta = \frac{R\underset{\sim}{\omega}(\underset{\sim}{\beta}\,|\,\underset{\sim}{\alpha})/(b\text{-}a)}{\mathrm{SSE}_{\underset{\sim}{\omega}}/(c\text{-}b)} \sim F_{b\text{-}a,\,c\text{-}b} \qquad (3.138)$$

under $H_0:\sigma_\beta^2 = 0$. We would reject H_0 at the ν level of significance if $F_\beta > F_{\nu,b-a,c-b}$ where $F_{\nu,b-a,c-b}$ is the upper $100\nu\%$ point of the corresponding F distribution. The fact that we reject H_0 for large values of our test statistic follows from the next lemma.

Lemma 3.4

a) $\mathrm{tr}[(I-P_1)H_1A_2A_2'H_1'] > 0$

b) $\mathrm{rank}[(I-P_1)H_1A_2A_2'H_1'] = b\text{-}a$, where $(I-P_1)H_1A_2A_2'H_1'$ is part of the expression for $(P_2-P_1)\big(\mathrm{Var}(\underset{\sim}{\omega})\big)$ in (3.137).

Proof:

a) Both $(I-P_1)$ and $H_1A_2A_2'H_1'$ are n.n.d. Thus $\mathrm{tr}[(I-P_1)H_1A_2A_2'H_1'] \geq 0$ $\big($Graybill (1983, p. 397)$\big)$. However, since $\mathrm{range}(P_1) \subset \mathrm{range}(P_2)$ $= \mathrm{range}(H_1A_2A_2'H_1')$ it follows that $(I-P_1)H_1A_2A_2'H_1' \neq 0$ $\big($if it were zero then $\mathrm{range}(H_1A_2A_2'H_1')$ would be contained in $\mathrm{range}(P_1)$, a contradiction$\big)$. Hence $\mathrm{tr}[(I-P_1)H_1A_2A_2'H_1'] > 0$ $\big($see Graybill (1983, p. 397)$\big)$.

b) First note that

$$P_2H_1A_2A_2'H_1' = H_1A_2A_2'H_1'.$$

Thus, since $P_1P_2 = P_1$

$$(I-P_1)H_1A_2A_2'H_1' = (P_2-P_1)H_1A_2A_2'H_1'.$$

Consequently,

$$\begin{aligned}\text{rank}[(I-P_1)H_1A_2A_2'H_1'] &= \text{rank}[(P_2-P_1)H_1A_2A_2'H_1'] \\ &\le \text{rank}(P_2-P_1) = \text{b-a}.\end{aligned}$$

Conversely, by the Frobenius Inequality (see Lemma A.1),

$$\begin{aligned}\text{rank}[(I-P_1)H_1A_2A_2'H_1'] &\ge \text{rank}(I-P_1) + \text{rank}(H_1A_2A_2'H_1') - \text{rank}(I_{c-1}) \\ &= \text{c-a} + \text{b-1} - (\text{c-1}) \\ &= \text{b-a}.\end{aligned}$$

Hence, $\text{rank}[(I-P_1)H_1A_2A_2'H_1'] = \text{b-a}$. □

If $\bar{\theta}_\beta(>0)$ denotes the average of the nonzero eigenvalues of $(I-P_1)H_1A_2A_2'H_1'$ then from (3.137)

$$E\left[\frac{R_{\underset{\sim}{\omega}}(\underset{\sim}{\beta}\,|\,\underset{\sim}{\alpha})}{\text{b-a}}\right] = \text{tr}\left[(P_2-P_1)\text{Var}(\underset{\sim}{\omega})\right]/(\text{b-a}) = \bar{\theta}_\beta\sigma_\beta^2 + \delta.$$

We note that $\bar{\theta}_\beta$ is an average since $\text{rank}\left[(I-P_1)H_1A_2A_2'H_1'\right] = \text{b-a}$ by (b) of this lemma. We now develop an exact test for σ_α^2.

3.2.4. An Exact Test Concerning σ_α^2

Introduction. In this section an exact test is constructed for the variance component associated with the nesting factor. Similarly to Section 3.2.3, we would like to use model (3.129) to develop our exact procedure. Unfortunately, because $\text{range}(B_1) \subset \text{range}(B_2)$ in (3.129), we have

$$\begin{aligned} R_{\underset{\sim}{\omega}}(\underset{\sim}{\alpha} \mid \underset{\sim}{\beta}) &= \underset{\sim}{\omega}'\left(I_{c-1} - B_2(B_2'B_2)^{-}B_2'\right)\underset{\sim}{\omega} - SSE_{\underset{\sim}{\omega}} \\ &= 0. \end{aligned}$$

Thus another approach is necessary. The problem here is similar to the problem we would have encountered if we had tried to use the original model (3.95) to construct an exact test for σ_β^2. Namely because $\text{range}(X_2) \subset \text{range}(X_3)$, $R(\underset{\sim}{\beta} \mid \mu, \underset{\sim}{\alpha}, \underset{\sim}{\gamma}) = 0$.

We overcame this problem by first averaging over the last subscript, then transforming the resulting model into one that had a zero mean vector. Finally, another transformation was made based on resampling from the error vector to arrive at model (3.129). From here we were able to derive an exact test for σ_β^2.

The same idea can be used again to develop an exact test for σ_α^2. Now, however, our starting model is model (3.129). This model is similar to an unbalanced random two-fold nested model (without an intercept term) in the sense that it enjoys all the same mathematical and statistical properties of such a model.

Initial development. Our first step will be to “average over the last subscript”. This can be accomplished by considering the $b \times 1$ random vector $\underset{\sim}{\bar{\omega}}$ defined as

$$\underset{\sim}{\bar{\omega}} = (B_2'B_2)^{-}B_2'\underset{\sim}{\omega}$$

$$= (A_2'H_1'H_1A_2)^{-}A_2'H_1'\underset{\sim}{\omega}$$

$$= (A_2'A_2)^{-1}A_2'H_1'\underset{\sim}{\omega}. \tag{3.139}$$

The last equality follows from Theorem 3.4(a).

Next we transform model (3.139) into one that has a “zero mean vector.” Although $E(\underset{\sim}{\bar{\omega}}) = \underset{\sim}{0}$, it is still convenient (for mathematical reasons) to make this transformation. Let H_1^* be the last b-1 rows of the $b \times b$ Helmert matrix. Then $H_1^*H_1^{*\prime} = I_{b-1}$ and $H_1^*\underset{\sim}{1}_b = \underset{\sim}{0}_{b-1}$. Define $\underset{\sim}{v}$ as the $(b-1) \times 1$ vector

$$\underset{\sim}{v} = H_1^*\underset{\sim}{\bar{\omega}}. \tag{3.140}$$

The mean and variance of $\underset{\sim}{v}$ is provided in the next lemma.

Lemma 3.5

a) $E(\underset{\sim}{v}) = \underset{\sim}{0}_{b-1}$

b) $$\mathrm{Var}(\underset{\sim}{v}) = H_1^*\Big(\bigoplus_{i=1}^{a} J_{b_i}\Big)H_1^{*\prime}\sigma_\alpha^2 + I_{b-1}\sigma_\beta^2 + H_1^*(A_2'A_2)^{-1}H_1^{*\prime}\delta \tag{3.141}$$

Proof:

a) From Theorem 3.3, $E(\underset{\sim}{\omega}) = \underset{\sim}{0}_{c-1}$. Thus

$$E(\underset{\sim}{v}) = H_1^* E(\bar{\underset{\sim}{\omega}})$$

$$= H_1^*(A_2'A_2)^{-1}A_2'H_1'E(\underset{\sim}{\omega}) = \underset{\sim}{0}.$$

b) Also from Theorem 3.3,

$$\mathrm{Var}(\underset{\sim}{\omega}) = H_1\tilde{A}_1\sigma_\alpha^2 + H_1\tilde{A}_2H_1'\sigma_\alpha^2 + \delta I_{c-1}.$$

Thus,

$$\mathrm{Var}(\underset{\sim}{v}) = H_1^*[\mathrm{Var}(\bar{\underset{\sim}{\omega}})]H_1^{*\prime}$$

$$= H_1^*(A_2'A_2)^{-1}A_2'H_1'[\mathrm{Var}(\underset{\sim}{\omega})]H_1A_2(A_2'A_2)^{-1}H_1^{*\prime}$$

$$= H_1^*(A_2'A_2)^{-1}A_2'H_1'[H_1\tilde{A}_1H_2'\sigma_\alpha^2 + H_1\tilde{A}_2H_1'\sigma_\beta^2 + \delta I_{c-1}]$$
$$\times H_1A_2(A_2'A_2)^{-1}H_1^{*\prime}$$

Now, from (3.107), $\tilde{A}_1 = A_2\left(\overset{a}{\underset{i=1}{\oplus}} J_{b_i}\right)A_2'$. Therefore

$$H_1\tilde{A}_1H_1' = H_1A_2\left(\overset{a}{\underset{i=1}{\oplus}} J_{b_i}\right)A_2'H_1'. \tag{3.142}$$

Also,

$$H_1^*(A_2'A_2)^{-1}A_2'H_1'H_1A_2 = H_1^*(A_2'A_2)^{-1}A_2'(I_c - \tfrac{1}{c}J_c)A_2$$

$$= H_1^*(A_2'A_2)^{-1}[A_2'A_2 - \tfrac{1}{c}A_2'J_cA_2]$$

$$= H_1^*(A_2'A_2)^{-1}[A_2'A_2 - \tfrac{1}{c}A_2'A_2J_bA_2'A_2] \quad \Big(\text{from } (3.107)\Big)$$

$$= H_1^*[I_b - \tfrac{1}{c}J_bA_2'A_2]$$

$$= H_1^*. \qquad (\text{recall that } H_1^*\underset{\sim}{1}_b = \underset{\sim}{0}) \qquad (3.143)$$

Consequently, from (3.142) and (3.143),

$$\mathrm{Var}(\underset{\sim}{y}) = [H_1^*\Big(\bigoplus_{i=1}^{a} J_{b_i}\Big)A_2'H_1'\sigma_\alpha^2 + H_1^*A_2'H_1'\sigma_\beta^2 + H_1^*(A_2'A_2)^{-1}A_2'H_1'\delta]H_1A_2(A_2'A_2)^{-1}H_1^{*\prime}$$

$$= H_1^*\Big(\bigoplus_{i=1}^{a} J_{b_i}\Big)H_1^{*\prime}\sigma_\alpha^2 + H_1^*H_1^{*\prime}\sigma_\beta^2 + H_1^*(A_2'A_2)^{-1}H_1^{*\prime}\delta$$

$$= H_1^*\Big(\bigoplus_{i=1}^{a} J_{b_i}\Big)H_1^{*\prime}\sigma_\alpha^2 + I_{b-1}\sigma_\beta^2 + H_1^*(A_2'A_2)^{-1}H_1^{*\prime}\delta. \qquad \square$$

Our last step will be to resample from the error vector. In this case, the error vector is $R^*\underset{\sim}{\omega}$ $\Big(R^* = I_{c-1} - P_2\Big)$ since the error sum of squares for the $\underset{\sim}{\omega}$ model $\Big($see (3.129)$\Big)$ is given by $SSE_{\underset{\sim}{\omega}} = \underset{\sim}{\omega}'(I_{c-1}-P_2)\underset{\sim}{\omega}$.

Utilizing resampling. Noting that R^* is symmetric, idempotent, and has rank c-b, we can write it as

$$R^* = C^*\Gamma^*C^{*\prime} \tag{3.144}$$

where C^* is orthogonal and Γ^* is diagonal whose first c-b diagonal entries are equal to unity and the remaining entries are equal to zero. Furthermore C^* and Γ^* can be partitioned as

$$\begin{gathered} C^* = [C_1^* : C_2^* : C_3^*] \\ \Gamma^* = \mathrm{diag}(I_{\eta_1^*}, I_{\eta_2^*}, 0), \end{gathered} \tag{3.145}$$

where

$$\begin{gathered} \eta_1^* = b-1 \\ \eta_2^* = c-2b+1. \end{gathered} \tag{3.146}$$

We note that $\eta_1^* + \eta_2^* = \mathrm{rank}(R^*)$ and $\eta_2^* > 0$ by (3.112)(iv). Also, C_1^*, C_2^*, and C_3^* are matrices of orders $(c-1)\times\eta_1^*$, $(c-1)\times\eta_2^*$, and $(c-1)\times\eta_1^*$, respectively, satisfying

$$\begin{gathered} C_i^{*\prime}C_i^* = I \qquad (i = 1,2,3) \\ C_i^{*\prime}C_j^* = 0 \qquad (i \neq j). \end{gathered} \tag{3.147}$$

Thus,

$$R^* = C_1^* C_1^{*\prime} + C_2^* C_2^{*\prime}. \tag{3.148}$$

Define the (b-1)×1 random vector $\underset{\sim}{\tau}$ as

$$\underset{\sim}{\tau} = \underset{\sim}{v} + (\lambda_{\max}^* I_{\eta_1^*} - L^*)^{\frac{1}{2}} C_1^{*\prime} \underset{\sim}{\omega} \tag{3.149}$$

where $\lambda_{\max}^*$ is the largest eigenvalue of $L^* = H_1^*(A_2^\prime A_2)^{-1} H_1^{*\prime}$ and $(\lambda_{\max}^* I_{\eta_1^*} - L^*)^{\frac{1}{2}}$ is a well-defined matrix with eigenvalues equaling the square roots of the eigenvalues of $(\lambda_{\max}^* I_{\eta_1^*} - L^*)$, which are nonnegative. The major result of this section is given in the following theorem.

Theorem 3.5

The random vector $\underset{\sim}{\tau}$ is normally distributed with a zero mean vector and a variance-covariance matrix given by

$$\mathrm{Var}(\underset{\sim}{\tau}) = H_1^* \Big(\bigoplus_{i=1}^{a} J_{b_i} \Big) H_1^{*\prime} \sigma_\alpha^2 + \delta^* I_{b-1} \tag{3.150}$$

where $\delta^* = \sigma_\beta^2 + \lambda_{\max}^* \delta$.

Proof:

Clearly $\underset{\sim}{\tau}$ is normally distributed. From Lemma 3.5 we know that $E(\underset{\sim}{v}) = \underset{\sim}{0}_{b-1}$. $E(\underset{\sim}{\omega}) = \underset{\sim}{0}_{c-1}$ from Theorem 3.3. Thus

$$E(\underset{\sim}{\tau}) = E(\underset{\sim}{v}) + (\lambda^*_{\max} I_{\eta_1^*} - L^*)^{\frac{1}{2}} C_1^{*\prime} E(\underset{\sim}{\omega}) \qquad \big(\text{see } (3.149)\big)$$

$$= \underset{\sim}{0}.$$

We claim that $\underset{\sim}{v} = H_1^* \underset{\sim}{\bar{\omega}} = (A_2'A_2)^{-1}A_2'H_1'\underset{\sim}{\omega}$ $\big($see (3.140) and (3.139)$\big)$ is independent of $C_1^{*\prime}\underset{\sim}{\omega}$ in (3.149). To verify this consider

$$(A_2'A_2)^{-1}A_2'H_1'[\mathrm{Var}(\underset{\sim}{\omega})]R^* = (A_2'A_2)^{-1}A_2'H_1'[\mathrm{Var}(\underset{\sim}{\omega})](I - P_2)$$

$$= \delta(A_2'A_2)^{-1}A_2'H_1'(I - P_2) \qquad \big(\text{recall } (3.127) \text{ and } (3.136)\big)$$

$$= 0.$$

The last equality follows since $\mathrm{range}\big[(A_2'A_2)^{-1}A_2'H_1'\big] = \mathrm{range}(P_2)$. Since $\mathrm{range}(C_1^*) \subset \mathrm{range}(R^*)$ it follows that

$$(A_2'A_2)^{-1}A_2'H_1[\mathrm{Var}(\underset{\sim}{\omega})]C_1^* = 0.$$

Thus $\underset{\sim}{v}$ is independent of $C_1^{*\prime}\underset{\sim}{\omega}$ as claimed. Consequently, from (3.149)

$$\mathrm{Var}(\underset{\sim}{\tau}) = \mathrm{Var}(\underset{\sim}{v}) + (\lambda^*_{\max} I_{\eta_1^*} - L^*)^{\frac{1}{2}} C_1^{*\prime}[\mathrm{Var}(\underset{\sim}{\omega})]C_1^*(\lambda^*_{\max} I_{\eta_1^*} - L^*)^{\frac{1}{2}} .$$

Now, recalling that $[\mathrm{Var}(\underset{\sim}{\omega})]R^* = \delta R^*$ $\big($see (3.127) and (3.136)$\big)$,

$$\begin{aligned} C_1^{*\prime}[\mathrm{Var}(\underset{\sim}{\omega})]C_1^* &= C_1^{*\prime}R^*[\mathrm{Var}(\underset{\sim}{\omega})]R^*C_1^* && \big(\text{from (3.148)}\big) \\ &= \delta C_1^{*\prime}R^*C_1^* && (R^* \text{ is idempotent}) \\ &= \delta C_1^{*\prime}C_1^* \\ &= \delta I_{b-1}. \end{aligned}$$

Hence, from Lemma 3.5(b) and the fact that $L^* = H_1^*(A_2'A_2)^{-1}H_1^{*\prime}$,

$$\begin{aligned} \mathrm{Var}(\underset{\sim}{\tau}) &= \mathrm{Var}(\underset{\sim}{v}) + (\lambda_{\max}^* I_{\eta_1^*} - L^*)\delta \\ &= H_1^*\Big(\overset{a}{\underset{i=1}{\oplus}} J_{b_i}\Big)H_1^{*\prime}\sigma_\alpha^2 + I_{b-1}\sigma_\beta^2 + L^*\delta + (\lambda_{\max}^* I_{\eta_1^*} - L^*)\delta \\ &= H_1^*\Big(\overset{a}{\underset{i=1}{\oplus}} J_{b_i}\Big)H_1^{*\prime}\sigma_\alpha^2 + I_{b-1}\delta^* \end{aligned}$$

where $\delta^* = \sigma_\beta^2 + \lambda_{\max}^*\delta$. □

We can infer from Theorem 3.5 that model (3.149) can be re-expressed as

$$\underset{\sim}{\tau} = \Big(H_1^* \overset{a}{\underset{i=1}{\oplus}} \underset{\sim}{1}_{b_i}\Big)\underset{\sim}{\alpha} + \underset{\sim}{\xi} \qquad (3.151)$$

where $\underset{\sim}{\xi} = H_1^*\underset{\sim}{\beta} + H_1^*\bar{\underset{\sim}{\epsilon}}^* + (\lambda_{\max}^* I_{\eta_1^*} - L^*)^{\frac{1}{2}}C_1^{*\prime}\underset{\sim}{\epsilon}^* \sim N(\underset{\sim}{0}, \delta^* I_{b-1})$ independently of $\underset{\sim}{\alpha}$ and $\bar{\underset{\sim}{\epsilon}}^* = (A_2'A_2)^{-1}A_2'H_1'\underset{\sim}{\epsilon}^*$. The vector $\underset{\sim}{\epsilon}^*$ is defined in (3.129). Model

(3.151) is similar to an unbalanced one-way random effects model with zero intercept. We note that

$$\operatorname{rank}\left(H_1^* \bigoplus_{i=1}^{a} \underset{\sim}{1}_{b_i}\right) \leq \operatorname{rank}\left(\bigoplus_{i=1}^{a} \underset{\sim}{1}_{b_i}\right) \leq a.$$

But, since $\left(\bigoplus_{i=1}^{a} \underset{\sim}{1}_{b_i}\right)\underset{\sim}{1}_a = \underset{\sim}{1}_b$ and $H_1^*\underset{\sim}{1}_b = \underset{\sim}{0}_{b-1}$, it follows that $\operatorname{rank}\left(H_1^* \bigoplus_{i=1}^{a} \underset{\sim}{1}_{b_i}\right) \leq a-1$. Conversely, by the Frobenius Inequality (see Lemma A.1),

$$\begin{aligned}\operatorname{rank}\left(H_1^* \bigoplus_{i=1}^{a} \underset{\sim}{1}_{b_i}\right) &\geq \operatorname{rank}(H_1^*) + \operatorname{rank}\left(\bigoplus_{i=1}^{a} \underset{\sim}{1}_{b_i}\right) - \operatorname{rank}(I_b) \\ &= b-1+a-b \\ &= a-1.\end{aligned}$$

Hence, $\operatorname{rank}\left(H_1^* \bigoplus_{i=1}^{a} \underset{\sim}{1}_{b_i}\right) = a-1$. We have therefore just proven Lemma 3.6.

Lemma 3.6

In (3.151), $\operatorname{rank}\left(H_1^* \bigoplus_{i=1}^{a} \underset{\sim}{1}_{b_i}\right) = a-1$.

We can now develop the exact test for σ_α^2.

The exact test. We now construct an exact test for $H_0\colon\sigma_\alpha^2 = 0$ vs. $H_a\colon\sigma_\alpha^2 \neq 0$. To start with let F be the (b-1) × a matrix

$$F = H_1^* \bigoplus_{i=1}^{a} \underset{\sim}{1}_{b_i}. \tag{3.152}$$

Also, let $\mathrm{SSE}_{\underset{\sim}{\tau}}$ denote the error sum of squares for model (3.151). That is,

$$\mathrm{SSE}_{\underset{\sim}{\tau}} = \underset{\sim}{\tau}'(I_{b-1} - F(F'F)^{-}F')\underset{\sim}{\tau}. \tag{3.153}$$

$R_{\tau}(\underset{\sim}{\alpha})$ will denote the sum of squares for the $\underset{\sim}{\alpha}$ effect in model (3.151). That is

$$R_{\tau}(\underset{\sim}{\alpha}) = \underset{\sim}{\tau}'F(F'F)^{-}F'\underset{\sim}{\tau}. \tag{3.154}$$

In the next theorem we find a generalized inverse for $F'F$ and show that $R_{\underset{\sim}{\tau}}(\underset{\sim}{\alpha})$ and $\mathrm{SSE}_{\underset{\sim}{\tau}}$ are independent and have scaled chi-squared distributions (the former having a chi-squared distribution only under $H_0:\sigma^2_\alpha = 0$).

Theorem 3.6

(a) A g-inverse of $F'F = \left(\overset{a}{\underset{i=1}{\oplus}} \underset{\sim}{1}'_{b_i}\right) H_1^{*\prime} H_1^{*} \left(\overset{a}{\underset{i=1}{\oplus}} \underset{\sim}{1}_{b_i}\right)$ is

$$(F'F)^{-} = \overset{a}{\underset{i=1}{\oplus}} b_i^{-1}.$$

(b) $R_{\underset{\sim}{\tau}}(\underset{\sim}{\alpha})/\delta^* \sim \chi^2_{a-1}$, under $H_0:\sigma^2_\alpha = 0$

(c) $\mathrm{SSE}_{\underset{\sim}{\tau}}/\delta^* \sim \chi^2_{b-a}$

(d) $R_{\underset{\sim}{\tau}}(\underset{\sim}{\alpha})$ is independent of $\mathrm{SSE}_{\underset{\sim}{\tau}}$.

Proof:

The proof of Theorem 3.6 follows along much the same reasoning as that for Theorem 3.4 and is thus omitted. □

A test statistic for testing $H_0:\sigma_\alpha^2=0$ vs. $H_a:\sigma_\alpha^2\neq 0$ is given by

$$F_\alpha=\frac{R_{\underset{\sim}{\tau}}(\underset{\sim}{\alpha})/(a-1)}{SSE_{\underset{\sim}{\tau}}/(b-a)}. \tag{3.155}$$

Under H_0, $F_\alpha \sim F_{a-1,b-a}$. Note that

$$E\left[\frac{R_{\underset{\sim}{\tau}}(\alpha)}{a-1}\right]=\overline{\theta}_\alpha\sigma_\alpha^2+\delta^* \tag{3.156}$$

where $\overline{\theta}_\alpha$ is the average of the eigenvalues of $FF'=H_1^*\left(\overset{a}{\underset{i=1}{\oplus}} J_{b_i}\right)H_1^{*\prime}$. If H^* denotes $b\times b$ Helmert matrix, then

$$H^*=\begin{bmatrix}\frac{1}{\sqrt{b}}\ \underset{\sim}{1}_b' \\ H_1^*\end{bmatrix}. \tag{3.157}$$

Consequently,

$$b=\sum_{i=1}^{a} b_i=\mathrm{tr}\left(\overset{a}{\underset{i=1}{\oplus}} J_{b_i}\right)=\mathrm{tr}\left(H^*\left(\overset{a}{\underset{i=1}{\oplus}} J_{b_i}\right)H^{*\prime}\right)$$

$$=\mathrm{tr}\begin{bmatrix}\frac{1}{b}\underset{\sim}{1}_b'\left(\overset{a}{\underset{i=1}{\oplus}} J_{b_i}\right)\underset{\sim}{1}_b & \frac{1}{\sqrt{b}}\underset{\sim}{1}_b'\left(\overset{a}{\underset{i=1}{\oplus}} J_{b_i}\right)H_1^{*\prime} \\ \frac{1}{\sqrt{b}}H_1^*\left(\overset{a}{\underset{i=1}{\oplus}} J_{b_i}\right)\underset{\sim}{1}_b & H_1^*\left(\overset{a}{\underset{i=1}{\oplus}} J_{b_i}\right)H_1^{*\prime}\end{bmatrix}$$

$$=\frac{1}{b}\underset{\sim}{1}_b'\left(\bigoplus_{i=1}^{a} J_{b_i}\right)\underset{\sim}{1}_b+\operatorname{tr}\left(H_1^*\left(\bigoplus_{i=1}^{a} J_{b_i}\right)H_1^{*\prime}\right)$$

$$=\frac{1}{b}\sum_{i=1}^{a} b_i^2+\operatorname{tr}\left(H_1^*\left(\bigoplus_{i=1}^{a} J_{b_i}\right)H_1^{*\prime}\right).$$

Hence,

$$\overline{\theta}_\alpha=\frac{1}{a-1}\left[b-\frac{1}{b}\sum_{i=1}^{a} b_i^2\right]>0. \tag{3.158}$$

Therefore we would reject $H_0\colon\sigma_\alpha^2=0$ for large values of F_α.

3.2.5. Reduction of the Exact Tests when the Design is Balanced

Introduction. We now show that the exact tests for σ_α^2 and σ_β^2 given in (3.155) and (3.138), respectively, reduce to the usual ANOVA tests when the design is balanced. The test statistic F_γ, given in Section 3.2.1, is the usual ANOVA test for testing $H_0\colon\sigma_\gamma^2=0$ vs. $H_a\colon\sigma_\gamma^2\neq 0$.

When the design is balanced we have

$$b_i\equiv b^*\ \forall i$$

$$c_{ij}\equiv c^*\ \forall(i,j) \tag{3.159}$$

$$n_{ijk}\equiv n^*\ \forall(i,j,k).$$

Consequently,

$$b = \sum_{i} b_i = ab^*$$

$$c = \sum_{i,j} c_{ij} = ab^*c^* \tag{3.160}$$

$$n = \sum_{i,j,k} n_{ijk} = ab^*c^*n^*$$

Also, recalling (3.104), (3.105), and (3.111)

$$A_1 = \bigoplus_{i} \underset{\sim}{1}_{b^*c^*} = I_a \otimes \underset{\sim}{1}_{b^*} \otimes \underset{\sim}{1}_{c^*}$$

$$A_2 = \bigoplus_{i,j} \underset{\sim}{1}_{c^*} = I_a \otimes \underset{\sim}{I}_{b^*} \otimes \underset{\sim}{1}_{c^*} \tag{3.161}$$

$$K = \bigoplus_{i,j,k} n^{*-1} = \frac{1}{n^*}(I_a \otimes I_{b^*} \otimes I_{c^*}).$$

Thus the matrix L defined in (3.115) becomes

$$L = H_1 K H_1' = \frac{1}{n^*} \; I_{c-1}. \tag{3.162}$$

It's largest eigenvalue is therefore $\lambda_{\max} = \frac{1}{n^*}$. The matrix L^*, used in (3.149), was defined as

$$L^* = H_1^* (A_2' A_2)^{-1} H_1^{*\prime}.$$

When the design is balanced, L^* becomes $\big($see (3.161)$\big)$

$$L^* = H_1^*(c^* I_{ab^*})^{-1} H_1^{*\prime}$$

$$= \frac{1}{c^*} I_{b-1} . \qquad (3.163)$$

The largest eigenvalue of L^*, $\lambda^*_{\max}$, is therefore $\frac{1}{c^*}$. The matrices P_1 and P_2 were defined in (3.136). When the design is balanced they reduce to

$$\begin{aligned} P_1 &= H_1(I_a \otimes \bar{J}_{b^*} \otimes \bar{J}_{c^*})H_1' \\ P_2 &= H_1(I_a \otimes I_{b^*} \otimes \bar{J}_{c^*})H_1' \end{aligned} \qquad (3.164)$$

where $\bar{J}_s = \frac{1}{s} \underset{\sim}{1}_s \underset{\sim}{1}_s'$.

Reduction of the exact tests. Consider first the vector $\underset{\sim}{\omega}$ defined in (3.126). When the design is balanced $\underset{\sim}{\omega}$ becomes

$$\underset{\sim}{\omega} = \underset{\sim}{u} + (\lambda_{\max} I_{\eta_1} - L)^{\frac{1}{2}} C_1' \underset{\sim}{y}$$

$$= \underset{\sim}{u} \qquad \big(\text{from } (3.162)\big)$$

$$= H_1 \bar{\underset{\sim}{y}} \qquad \big(\text{see } (3.113)\big). \qquad (3.165)$$

Using (3.163), the vector $\underset{\sim}{\tau}$, defined in (3.149), reduces to

$$\begin{aligned}
\underset{\sim}{\tau} &= \underset{\sim}{v} + (\lambda_{\max}^{*} I_{\eta_1^*} - L^*)^{\frac{1}{2}} C_1^{*\prime} \underset{\sim}{\omega} \\
&= \underset{\sim}{v} && \Big(\text{from } (3.163)\Big) \\
&= H_1^* \underset{\sim}{\bar{\omega}} && \Big(\text{see } (3.140)\Big) \\
&= H_1^* (A_2' A_2)^{-1} A_2' H_1' \underset{\sim}{\omega} && \Big(\text{see } (3.139)\Big) \\
&= \frac{1}{c^*} H_1^* (I_a \otimes I_{b^*} \otimes \underset{\sim}{1}_{c^*}') H_1' \underset{\sim}{\omega} && \Big(\text{from } (3.161)\Big). \quad (3.166)
\end{aligned}$$

Since the design is balanced,

$$I_{c-1} - P_2 = H_1\Big[I_a \otimes I_{b^*} \otimes (I_{c^*} - \bar{J}_{c^*})\Big]H_1'$$

and

$$P_2 - P_1 = H_1\Big[I_a \otimes (I_{b^*} - \bar{J}_{b^*}) \otimes \bar{J}_{c^*}\Big]H_1' .$$

Consequently,

$$\begin{aligned}
R_{\underset{\sim}{\omega}}(\underset{\sim}{\beta} \mid \underset{\sim}{\alpha}) &= \underset{\sim}{\omega}'(P_2 - P_1)\underset{\sim}{\omega} && \Big(\text{see } (3.135)\Big) \\
&= \underset{\sim}{\bar{y}}' H_1' H_1\Big[I_a \otimes (I_{b^*} - \bar{J}_{b^*}) \otimes \bar{J}_{c^*}\Big]H_1' H_1 \underset{\sim}{\bar{y}}
\end{aligned}$$

$$= \underset{\sim}{y}'\left[I_a \otimes (I_{b^*} - \bar{J}_{b^*}) \otimes \bar{J}_{c^*}\right]\underset{\sim}{y}. \qquad (3.167)$$

The last equality follows since $H_1'H_1 = I_c - \bar{J}_c$ and $\bar{J}_{b^*}(I_{b^*} - \bar{J}_{b^*}) = 0$. Also

$$\mathrm{SSE}_{\underset{\sim}{\omega}} = \underset{\sim}{\omega}'(I_{c-1} - P_2)\underset{\sim}{\omega} \qquad \Big(\text{see } (3.132)\Big)$$

$$= \underset{\sim}{y}'H_1'H_1\left[I_a \otimes I_{b^*} \otimes (I_{c^*} - \bar{J}_{c^*})\right]H_1'H_1\underset{\sim}{y}$$

$$= \underset{\sim}{y}'\left[I_a \otimes I_{b^*} \otimes (I_{c^*} - \bar{J}_{c^*})\right]\underset{\sim}{y}. \qquad (3.168)$$

The last equality follows since $\bar{J}_{c^*}(I_{c^*} - \bar{J}_{c^*}) = 0$. Furthermore, from (3.153) and (3.154) we have

$$\mathrm{SSE}_{\underset{\sim}{\tau}} = \underset{\sim}{\tau}'(I_{b-1} - F(F'F)^{-}F')\underset{\sim}{\tau}$$

$$= \underset{\sim}{\tau}'(I_{b-1} - \frac{1}{b^*}FF')\underset{\sim}{\tau} \qquad \Big(\text{by Theorem 3.6(a)}\Big)$$

$$= \underset{\sim}{\tau}'(I_{b-1} - H_1^*(I_a \otimes \bar{J}_{b^*})H_1^{*\prime})\underset{\sim}{\tau} \qquad \Big(\text{see } (3.152)\Big)$$

$$= \underset{\sim}{\tau}'H_1^*\left[I_a \otimes (I_{b^*} - \bar{J}_{b^*})\right]H_1^{*\prime}\underset{\sim}{\tau}.$$

From (3.166) we can further reduce $\mathrm{SSE}_{\underset{\sim}{\tau}}$ to

$$\mathrm{SSE}_{\underset{\sim}{\tau}} = \frac{1}{c^{*2}}\underset{\sim}{\omega}'H_1Q_\tau'H_1^{*\prime}H_1^*\left[I_a \otimes (I_{b^*} - \bar{J}_{b^*})\right]H_1^{*\prime}H_1^*Q_\tau H_1'\underset{\sim}{\omega}$$

$$= \frac{1}{c^{*2}} \underset{\sim}{\omega}' H_1 Q_\tau' \left[I_a \otimes (I_{b^*} - \bar{J}_{b^*}) \right] Q_\tau H_1' \underset{\sim}{\omega}$$

where, from (3.166),

$$Q_\tau = I_a \otimes I_{b^*} \otimes \underset{\sim}{1}_{c^*}'.$$

Since

$$\frac{1}{c^*} Q_\tau' \left[I_a \otimes (I_{b^*} - \bar{J}_{b^*}) \right] Q_\tau = I_a \otimes (I_{b^*} - \bar{J}_{b^*}) \otimes \bar{J}_{c^*}$$

we have

$$SSE_{\underset{\sim}{\tau}} = \frac{1}{c^*} \underset{\sim}{\omega}' H_1 \left[I_a \otimes (I_{b^*} - \bar{J}_{b^*}) \otimes \bar{J}_{c^*} \right] H_1' \underset{\sim}{\omega}$$

$$= \frac{1}{c^*} \underset{\sim}{\omega}' (P_2 - P_1) \underset{\sim}{\omega}.$$

Thus,

$$c^* SSE_{\underset{\sim}{\tau}} = R_{\underset{\sim}{\omega}}(\underset{\sim}{\beta} \,|\, \underset{\sim}{\alpha}). \qquad (3.169)$$

Similarly, we can show that

$$R_\tau(\underset{\sim}{\alpha}) = \underset{\sim}{\tau}' F (F'F)^- F' \underset{\sim}{\tau}$$

$$= \frac{1}{c^*} \underset{\sim}{\omega}' H_1 (I_a \otimes \bar{J}_{b^*} \otimes \bar{J}_{c^*}) H_1' \underset{\sim}{\omega}$$

$$= \frac{1}{c^*} \underset{\sim}{\omega}' P_1 \underset{\sim}{\omega}.$$

Thus,

$$c^*R_{\underset{\sim}{\mathcal{T}}}(\underset{\sim}{\alpha}) = \bar{\underset{\sim}{y}}'H_1'P_1H_1\bar{\underset{\sim}{y}}. \qquad (3.170)$$

Since the design is balanced,

$$\bar{\underset{\sim}{y}} = \frac{1}{n^*}(I_a \otimes I_{b^*} \otimes I_{c^*} \otimes \underset{\sim}{1}'_{n^*})\underset{\sim}{y}. \qquad (3.171)$$

Thus (3.167), (3.168), and (3.170) can be expressed as, respectively,

$$R_{\underset{\sim}{\omega}}(\underset{\sim}{\beta}\,|\,\underset{\sim}{\alpha}) = \frac{1}{n^*}\underset{\sim}{y}'\left[I_a \otimes (I_{b^*} - \bar{J}_{b^*}) \otimes \bar{J}_{c^*} \otimes \bar{J}_{n^*}\right]\underset{\sim}{y} \qquad (3.172)$$

$$SSE_{\underset{\sim}{\omega}} = \frac{1}{n^*}\underset{\sim}{y}'\left[I_a \otimes I_{b^*} \otimes (I_{c^*} - \bar{J}_{c^*}) \otimes \bar{J}_{n^*}\right]\underset{\sim}{y} \qquad (3.173)$$

and

$$R_{\underset{\sim}{\mathcal{T}}}(\underset{\sim}{\alpha}) = \frac{1}{n^*c^*}\left[(I_a - \bar{J}_a) \otimes \bar{J}_{b^*} \otimes \bar{J}_{c^*} \otimes \bar{J}_{n^*}\right]\underset{\sim}{y}. \qquad (3.174)$$

Therefore, from Khuri (1982), it follows that $n^*c^*R_{\underset{\sim}{\mathcal{T}}}(\underset{\sim}{\alpha})$, $n^*R_{\underset{\sim}{\omega}}(\underset{\sim}{\beta}\,|\,\underset{\sim}{\alpha}) = n^*c^*SSE_{\underset{\sim}{\mathcal{T}}}$, and $n^*SSE_{\underset{\sim}{\omega}}$ reduce to the usual ANOVA sums of squares for balanced data.

3.2.6. A Numerical Example

The following data are taken from Bennett and Franklin (1954, p. 405) and has been made unbalanced for illustrative purposes. The data represent a part of a statistical study on the variability of a number of properties of crude smoked rubber. The figures given in Table 3.2 represent measurements of the modulus at 700% elongation made using 24 samples. On each of these samples at most 3 determinations of the modulus at 700% elongation were made, resulting in a total of 53 observations. The three factors of interest are supplier, batch, and mix. We assume each factor to be random. $Y_{ijk\ell}$ will represent the ℓ^{th} replicate determination from mix k, which is derived from batch j of supplier i. The linear model in this experiment is

$$y_{ijk\ell}=\mu+\alpha_i+\beta_{ij}+\gamma_{ijk}+\epsilon_{ijk\ell},$$

where $i=1,2,3,4$; $j=1,2,\cdots,b_i$; $k=1,2$; $\ell=1,2,\cdots,n_{ijk}$. In this case $b_1=2$, $b_2=4$, $b_3=3$, and $b_4=3$. Notice that $c_{ij}=2$ for all (i,j) so that the design associated with this experiment is partially balanced.

We start our analysis by testing for a mix effect. That is, we test for $H_0:\sigma_\gamma^2=0$ vs $H_a:\sigma_\gamma^2\neq 0$. The appropriate test statistic is F_γ given in Section 3.2.1. In the present case we find

$F_\gamma = 1.3110$
numerator d.f. $= 12$
denominator d.f. $= 29$
p-value $= 0.2650$.

Thus there does not appear to be any significant variation due to the mix effect.

To test for a significant batch effect we use F_β, the test statistic developed in Section 3.2.3. For these data we find

$$F_\beta = 0.3800$$

numerator d.f. $= 8$

denominator d.f. $= 12$

p-value $= 0.9114$.

No significant variation in the moduli of the 24 samples exist among batches.

The test statistic F_α $\big($see (3.155)$\big)$ is used to test for a significant supplier effect. We find

$$F_\alpha = 39.0024$$

numerator d.f. $= 3$

denominator d.f. $= 8$

p-value < 0.0001.

There is highly significant variation in the moduli of the samples among the four suppliers.

All calculations were performed using S-plus.

Table 3.2 An example of an unbalanced random 3-fold nested model

Supplier	A		B		C		D	
Mix	1	2	1	2	1	2	1	2
Batch I	196	226	196	196	189	196	323	262
	200		186	156	186	175	279	234
			190				251	
Batch II	250	248	193	186	211	196	238	262
	238	249		196	197	190	250	272
					210			
Batch III			204	174	196	180	273	223
			165	172	197	166	221	256
			194		186			230
Batch IV			209	202				
			221	211				
				204				

(Source: Bennett and Franklin (1954, p. 405))

3.2.7. The Power of the Exact Tests

Introduction. The approximate power of the exact test concerning σ_α^2, the variance component associated with the nesting factor, is derived in this section. Similar expressions can also be developed for the other two variance components. We again use Hirotsu's (1979) results to develop our expression for the power (see Appendix D).

Let Ψ_α denote the power of the test for σ_α^2. Then from (3.155)

$$\Psi_\alpha = P\left[\frac{R_{\underset{\sim}{\tau}}(\underset{\sim}{\alpha})/(a-1)}{SSE_{\underset{\sim}{\tau}}/(b-a)} \geq F_{\nu,a-1,b-a} \mid H_a\right] \tag{3.175}$$

where $H_a:\sigma_\alpha^2 \neq 0$ and $R_{\underset{\sim}{\tau}}(\underset{\sim}{\alpha})$ and $SSE_{\underset{\sim}{\tau}}$ are defined in, respectively, (3.154) and (3.153). Under H_a, $R_{\underset{\sim}{\tau}}(\underset{\sim}{\alpha})$ is distributed like a random variable of the form (see Box(1954b))

$$\Upsilon_\alpha = \sum_{j=1}^{a-1} \theta_{\alpha j}\chi_j \tag{3.176}$$

where the χ_j's are independent chi-squared variates with one degree of freedom, and $\theta_{\alpha j}$ is the j^{th} nonzero eigenvalue of

$$U = FF'\sigma_\alpha^2 + F(F'F)^{-}F'\delta^*, \tag{3.177}$$

where F is defined in (3.152). Equation (3.177) follows from Theorem 3.5. Clearly FF' and $F(F'F)^{-}F'$ commute and $\text{rank}(FF':F(F'F)^{-}F')$

$=\text{rank}(FF') = a-1$. Thus there exists an orthogonal matrix P such that

$$PUP' = \begin{pmatrix} \Lambda_\alpha & 0 \\ 0 & 0 \end{pmatrix}\sigma_\alpha^2 + \begin{pmatrix} I_{a-1} & 0 \\ 0 & 0 \end{pmatrix}\delta^* \tag{3.178}$$

where Λ_α is the $(a-1)\times(a-1)$ diagonal matrix whose diagonal entries are the nonzero eigenvalues of FF′. (If PUP′ is not in the form indicated in (3.178) then pre- and post-multiply by permutation matrices until this form is achieved. The locations along the diagonal of the nonzero eigenvalues of FF′ and $F(F'F)^- F'$ must match exactly.) The j^{th} nonzero eigenvalue of U, $\theta_{\alpha j}$, is therefore

$$\theta_{\alpha j} = \lambda_{\alpha j}\sigma_\alpha^2 + \delta^*$$

where $\lambda_{\alpha j}$ is the j^{th} diagonal entry of Λ_α $(j = 1,2,\cdots,a-1)$.

The approximate power. To obtain the approximate power (see Appendix D) let

$$MSE_{\underset{\sim}{T}} = SSE_{\underset{\sim}{T}}/(b-a) \tag{3.179}$$

$$H = \frac{R_{\underset{\sim}{T}}(\underset{\sim}{\alpha})/df}{MSE_{\underset{\sim}{T}}/\delta^*} \tag{3.180}$$

$$h = \frac{\delta^*(a-1)}{d^* f} F_{\nu,a-1,b-a}. \tag{3.181}$$

If $\xi_\alpha = \frac{\sigma_\alpha^2}{\delta^*} = \frac{\sigma_\alpha^2}{\sigma_\beta^2 + \lambda_{\max}^*\delta}$, then in this case

$$d^* = \delta^* \frac{\sum_{j=1}^{a-1} (\lambda_{\alpha j}\xi_\alpha + 1)^2}{\sum_{j=1}^{a-1} (\lambda_{\alpha j}\xi_\alpha + 1)} \tag{3.182}$$

$$f = \frac{\left[\sum_{j=1}^{a-1} (\lambda_{\alpha j}\xi_\alpha + 1)\right]^2}{\sum_{j=1}^{a-1} (\lambda_{\alpha j}\xi_\alpha + 1)^2} \tag{3.183}$$

Furthermore, define

$$f_2 = b\text{-}a$$

$$\rho = \frac{\left[\sum_{j=1}^{a-1} (\lambda_{\alpha j}\xi_\alpha + 1)\right]\left[\sum_{j=1}^{a-1} (\lambda_{\alpha j}\xi_\alpha + 1)^3\right]}{\left[\sum_{j=1}^{a-1} (\lambda_{\alpha j}\xi_\alpha + 1)^2\right]^2} - 1. \tag{3.184}$$

Then Ψ_α $\big($in (3.175)$\big)$ can be approximated by using the formula given in (3.86) and making the appropriate substitutions for H, h, d*, f, f_2, and ρ. We note that "d*" in (3.182) plays the roll of "c" in (3.86).

Since $\delta = \sigma_\gamma^2 + \lambda_{\max}\sigma_\epsilon^2$ we see that the power of the exact test will depend on the design parameters $\lambda_{\max}$, $\lambda_{\max}^*$, $\lambda_{\alpha 1}, \cdots, \lambda_{\alpha,a-1}$; the level of significance; and the ratios $\sigma_\alpha^2/\sigma_\epsilon^2$, $\sigma_\beta^2/\sigma_\epsilon^2$, and $\sigma_\gamma^2/\sigma_\epsilon^2$ of the variance components.

The design parameters $\lambda_{\max}$ and $\lambda^*_{\max}$. Bounds for $\lambda_{\max}$ and $\lambda^*_{\max}$ are given in the next lemma.

Lemma 3.7

(a) The largest eigenvalue, $\lambda_{\max}$, of the matrix L in (3.115) satisfies the double inequality

$$\frac{1}{c} \sum_{i,j,k} \frac{1}{n_{ijk}} \leq \lambda_{\max} \leq \frac{1}{n^{(1)}} \tag{3.185}$$

where $n^{(1)} = \min_{i,j,k} (n_{ijk})$.

(b) The largest eigenvalue, $\lambda^*_{\max}$, of the matrix $L^* = H^*_1(A'_2A_2)^{-1}H^{*\prime}_1$ (see (3.149)) satisfies the double inequality

$$\frac{1}{b} \sum_{i,j} \frac{1}{c_{ij}} \leq \lambda^*_{\max} \leq \frac{1}{c^{(1)}} \tag{3.186}$$

where $c^{(1)} = \min_{i,j} (c_{ij})$.

Proof:

(a) If $e_{\max}(A)$ denotes the largest eigenvalue of the matrix A, then

$$\lambda_{\max} = e_{\max}(L) = e_{\max}(H_1KH'_1)$$

$$= e_{\max}(KH'_1H_1)$$

$$\le e_{\max}(K)e_{\max}(H_1'H_1)$$

$$= e_{\max}(K)e_{\max}(H_1H_1')$$

$$= e_{\max}(K) \qquad (H_1 \text{ has orthonormal rows})$$

$$= \frac{1}{n^{(1)}}. \tag{3.187}$$

It is also true that $\lambda_{\max}$ is greater than or equal to the average eigenvalue of L. That is,

$$\lambda_{\max} \ge \frac{1}{c-1}\mathrm{tr}(L) = \frac{1}{c-1}\mathrm{tr}(H_1KH_1')$$

$$= \frac{1}{c-1}\mathrm{tr}[K(I_c - \tfrac{1}{c}J_c)]$$

$$= \frac{1}{c-1}[\mathrm{tr}(K) - \tfrac{1}{c}\mathrm{tr}(KJ_c)]$$

$$= \frac{1}{c-1}[\mathrm{tr}(K) - \tfrac{1}{c}\underset{\sim}{1}_c'K\underset{\sim}{1}_c]$$

$$= \frac{1}{c-1}\left[\sum_{i,j,k}\frac{1}{n_{ijk}} - \frac{1}{c}\sum_{i,j,k}\frac{1}{n_{ijk}}\right]$$

$$= \frac{1}{c}\sum_{i,j,k}\frac{1}{n_{ijk}} \tag{3.188}$$

Thus, from (3.187) and (3.188),

$$\frac{1}{c}\sum_{i,j,k}\frac{1}{n_{ijk}} \le \lambda_{\max} \le \frac{1}{n^{(1)}}.$$

(b) Recalling that $(A_2'A_2)^{-1} = \underset{i,j}{\oplus} c_{ij}^{-1}$ and H_1^* has orthonormal rows (see (3.105) and (3.140)), we see that the proof of (b) will follow similarly to that of (a). □

The value of $\lambda_{\max}$ will not depend on which (c-1)×c suborthogonal matrix is used in (3.113) (see also (3.115)). To see this let $\tilde{H}$ be any other c×c orthogonal matrix whose first row is $\frac{1}{\sqrt{c}}\, \underset{\sim}{1}_c'$. Let $\tilde{H}_1$ be the (c-1)×c matrix resulting from the deletion of the first row of $\tilde{H}$. Then there exists a (c-1)×(c-1) orthogonal matrix M such that

$$\tilde{H}_1 = MH_1 .$$

Consequently,

$$\begin{aligned} e_{\max}(\tilde{H}_1K\tilde{H}_1') &= e_{\max}(MH_1KH_1'M') \\ &= e_{\max}(H_1KH_1') \qquad (M'M = I_{c-1}) \\ &= \lambda_{\max} . \end{aligned}$$

Similarly, $\lambda_{\max}^*$ is invariant to the choice of which (b-1)×b suborthogonal matrix is used in (3.140).

3.2.8 Concluding Remarks

As long as two rank conditions are satisfied, the results given in Seely and El-Bassiouni's (1983) Section 3, allow us to conclude that an exact test for the second to last stage effect in unbalanced nested designs can be constructed by comparing the mean square for that effect (adjusting for all the other effects in the model) to the error mean square for the model. Such a test would be a Wald's test. Thus, the test for σ_γ^2 given in Section 3.2.1 is a Wald's test.

The above observation provided the motivation for the two transformations introduced in (3.176) and (3.149). Each of these transformations was based on resampling from the error vector. The first reduced the analysis of the unbalanced 3-fold nested random model to that of an unbalanced 2-fold nested random model. While the second reduced the problem to that of analyzing an unbalanced 1-fold nested random model. In either case, the exact test was based on comparing the mean square associated with the second to last stage effect in the new (transformed) model to the error mean square associated with the new model. Consequently, both of these exact tests are Wald's tests.

Although not pursued in this dissertation, the extension to more general unbalanced nested models should offer no resistance. A partial extension is examined in Section 3.4. There we study a general unbalanced random nested model where the imbalance affects the last two stages only.

3.3 The General Unbalanced Mixed Model with Imbalance Affecting the Last Stage Only

3.3.1 Introduction

Khuri (1990) developed exact tests for testing hypotheses concerning the variance components in general unbalanced random models where the imbalance affected the last stage only. This extended the work of Khuri and Littell (1987). Gallo and Khuri (1990) constructed exact tests for both the random- and fixed-effects in an unbalanced mixed two-way cross-classification model. In this section we extend the work of Gallo and Khuri (1990) by considering a general unbalanced mixed model. We will require that the imbalance affects the last stage only.

After some preliminary development we start our analysis by studying the random effects in the model. Next we construct procedures for the fixed-effects. Finally, we reexamine our model assuming only a few cell counts are large relative to the smallest cell frequency. Under this scenario we will employ the results in Khuri (1984) to construct exact simultaneous confidence intervals on estimable linear functions of the fixed-effects parameters.

In our construction of the exact procedures we will closely follow the terminology and notation laid out in Zyskind (1960), Smith and Hocking (1978), and Khuri (1982, 1984, 1990).

3.3.2 Preliminary Development

Introduction. A mixed model for a general unbalanced design whose imbalance arises from unequal cell frequencies in the last stage only can be written in the form

$$y_\theta = \sum_{i=0}^{\nu} \gamma_{\theta_i(\bar{\theta}_i)} + \epsilon_\theta \tag{3.189}$$

where $\theta = (k_1, k_2, \cdots, k_s)$ is a complete set of subscripts that identify a typical response, y. The sets θ_i and $\bar{\theta}_i$ are the sets of nonrightmost and rightmost bracket subscripts (see Khuri (1982)), respectively, associated with the i^{th} effect $\gamma_{\theta_i(\bar{\theta}_i)} (i = 0, 1, \cdots, \nu)$. When $i = 0$ we define $\theta_0 = \bar{\theta}_0 = \phi$, the empty set, and the corresponding γ is the grand mean. We assume that $\gamma_{\theta_i(\bar{\theta}_i)}$, for $i = 0, 1, \cdots, \nu\text{-}p$ $(0 < p < \nu)$, are unknown parameters while $\gamma_{\theta_i(\bar{\theta}_i)}$, for $i = \nu\text{-}p+1, \cdots, \nu$, and ϵ_θ are independent, normally distributed random variables with zero means and variances $\sigma^2_{\nu\text{-}p+1}, \cdots, \sigma^2_\nu$, and σ^2_ϵ, respectively.

Since the design is balanced except for its last stage, the ranges of subscripts $k_1, k_2, \cdots, k_s$ can be expressed in the form

$$k_j = \begin{cases} 1, 2, \cdots, a_j & \text{for } j=1,2,\cdots,s\text{-}1 \\ 1, 2, \cdots, n_\tau & \text{for } j=s, \end{cases} \tag{3.190}$$

where $\tau = (k_1, k_2, \cdots, k_{s-1})$ consists of the first s-1 subscripts of θ. We assume that the data set contains no missing observations.

Let us define the set T by

$$T=\{\tau=(k_1,k_2,\cdots,k_{s-1}):\ k_j=1,2,\cdots,a_j;\ j=1,2,\cdots,s-1\}. \tag{3.191}$$

Then c, the number of elements in T, is

$$c=\prod_{j=1}^{s-1} a_j. \tag{3.192}$$

Also, let $\psi_i=(\theta_i:\overline{\theta}_i)$ be the set of subscripts associated with the i^{th} effect which results from concatenating the sets of θ_i and $\overline{\theta}_i$ $(i=0,1,\cdots,\nu)$; ψ_i^c is the complement of ψ_i with respect to τ $(i=0,1,\cdots,\nu)$. The complement is with respect to τ and not θ since k_s only appears in the ϵ_θ term of (3.189).

Matrix formulation. In matrix notation model (3.189) can be expressed as

$$\underset{\sim}{y}=\sum_{i=0}^{\nu} X_i\underset{\sim}{\beta}_i+\underset{\sim}{\epsilon}, \tag{3.193}$$

where $\underset{\sim}{y}$ is the $n\times 1$ vector of observations ($n=$ total number of observations $=\sum_{\tau\in T} n_\tau$); X_i is an $n\times c_i$ $(i=0,1,\cdots\nu)$ matrix of ones and zeros; $\underset{\sim}{\beta}_i$ is a vector consisting of the c_i elements of $\gamma_{\theta_i(\overline{\theta}_i)}$ $(i=0,1,\cdots,\nu)$; and $\underset{\sim}{\epsilon}$ is the $n\times 1$ vector of random errors. The integer c_i is given by

$$c_i=\begin{cases} 1 & \text{for } i=0 \\ \prod_{k_j\epsilon\psi_i} a_j & \text{for } i=1,2,\cdots,\nu. \end{cases} \tag{3.194}$$

The assumptions made earlier concerning the $\gamma_{\theta_i}(\bar{\theta}_i)$ $(i=0,1,\cdots,\nu)$ can now be restated as

(i) $\underset{\sim}{\beta}_0, \underset{\sim}{\beta}_1, \cdots, \underset{\sim}{\beta}_{\nu-p}$ are unknown parameter vectors for $0 < p < \nu$.

(ii) The vectors $\underset{\sim}{\beta}_{\nu-p+1}, \cdots, \underset{\sim}{\beta}_\nu$, and $\underset{\sim}{\epsilon}$ are independent and normally distributed random vectors with zero mean vectors and variance-covariance matrices given by

$$\mathrm{Var}(\underset{\sim}{\beta}_i) = \sigma_i^2 I_{c_i} \qquad \text{for } i = \nu\text{-}p+1, \cdots, \nu$$

$$\mathrm{Var}(\underset{\sim}{\epsilon}) = \sigma_\epsilon^2 I_n .$$

Besides these assumptions we will also assume

(iii) the model used is model (3.189) $\Big($or model (3.193)$\Big)$

(iv) $$n > \max\Big(2c - \sum_{i=0}^{\nu-p} m_i,\ 2c - \sum_{i \in \mathcal{S}^c} m_i,\ 2c - \sum_{i \in \mathcal{T}^c} m_i\Big), \tag{3.195}$$
where m_i, $\mathcal{S}^c$, and $\mathcal{T}^c$ will be defined later.

Assumption (iv) (along with the other assumptions) are needed to insure that our procedures are valid. Later we will strengthen Assumption (iv) by assuming that $n > 2c$ $\Big($see Section (3.3.7)$\Big)$.

From Assumptions (i) and (ii) it follows that model (3.193) can be written as

$$\underset{\sim}{y} = X\underset{\sim}{g} + Z\underset{\sim}{h} + \underset{\sim}{\epsilon} \tag{3.196}$$

where $X = (X_0 : X_1 : \cdots : X_{\nu-p})$; $Z = (X_{\nu-p+1} : X_{\nu-p+2} : \cdots : X_\nu)$; $\underset{\sim}{g} = (\underset{\sim}{\beta}'_0 : \underset{\sim}{\beta}'_1 : \cdots : \underset{\sim}{\beta}'_{\nu-p})'$ is the vector of fixed effects and $\underset{\sim}{h} = (\underset{\sim}{\beta}'_{\nu-p+1} : \underset{\sim}{\beta}'_{\nu-p+2} : \cdots : \underset{\sim}{\beta}'_\nu)'$ is the vector of random effects. It is easy to verify that

$$\begin{aligned} E(\underset{\sim}{y}) &= X\underset{\sim}{g} \\ \Omega = \mathrm{Var}(\underset{\sim}{y}) &= \sum_{i=\nu-p+1}^{\nu} \sigma_i^2 X_i X_i' + \sigma_\epsilon^2 I_n . \end{aligned} \tag{3.197}$$

Cell means. If it were not for the $\underset{\sim}{\epsilon}$ term, which consists of the n elements of ϵ_θ, model (3.193) or (3.196) would represent a general balanced mixed-effects model. Various results concerning general balanced mixed models can be found in, for example, Smith and Hocking (1978) and Khuri (1982, 1984). The idea then is to reduce the analysis of the unbalanced mixed model to that of a balanced model. To that end let

$$\bar{y}_\tau = \frac{1}{n_\tau} \sum_{k_s=1}^{n_\tau} y_\theta, \quad \tau \in T \tag{3.198}$$

if $\underset{\sim}{\bar{y}}$ is the vector consisting of the c values of $\bar{y}_\tau$, $\tau \in T$, then, from (3.198), we have

$$\underset{\sim}{\bar{y}} = D\underset{\sim}{y} \tag{3.199}$$

where

$$D = (W'W)^{-1}W'$$

and W is the $n \times c$ block diagonal matrix

$$W = \underset{\tau \in T}{\oplus} \mathbf{1}_{n_\tau}.$$

Clearly,

$$D = \underset{\tau \in T}{\oplus} \frac{1}{n_\tau}\mathbf{1}'_{n_\tau}. \tag{3.200}$$

Now, from (3.189) and (3.198) we have

$$\bar{y}_\tau = \sum_{i=1}^{\nu} \gamma_{\theta_i(\bar{\theta}_i)} + \bar{\epsilon}_\tau, \quad \tau \in T \tag{3.201}$$

where

$$\bar{\epsilon}_\tau = \frac{1}{n_\tau} \sum_{k_s=1}^{n_\tau} \epsilon_\theta, \quad \tau \in T.$$

Since $\sum_{i=0}^{\nu} \gamma_{\theta_i(\bar{\theta}_i)}$ in (3.201) is of the same form as in a balanced model, we can express (3.201) in matrix form as (see Khuri (1982))

$$\bar{\mathbf{y}}_\tau = \sum_{i=0}^{\nu} H_i \boldsymbol{\beta}_i + \bar{\boldsymbol{\epsilon}} \tag{3.202}$$

where the $c \times c_i$ matrix H_i can be expressed as

$$H_i = \overset{s-1}{\underset{\ell=1}{\otimes}} L_{i\ell}, \qquad i = 0, 1, \cdots, \nu \tag{3.203}$$

and $L_{i\ell}$ is given by

$$L_{i\ell} = \begin{cases} I_{a_\ell} & \text{for } k_\ell \in \psi_i \\ \underset{\sim}{1}_{a_\ell} & \text{for } k_\ell \in \psi_i^c \end{cases} \tag{3.204}$$

$(i = 0, 1, \cdots, \nu;\ \ell = 0, 1, \cdots, s-1)$. Furthermore, if

$$\begin{aligned} H &= (H_0 : H_1 : \cdots : H_{\nu-p}) \\ F &= (H_{\nu-p+1} : H_{\nu-p+2} : \cdots : H_\nu) \end{aligned} \tag{3.205}$$

then from (3.202) we obtain

$$\bar{\underset{\sim}{y}} = H\underset{\sim}{g} + F\underset{\sim}{h} + \bar{\underset{\sim}{\epsilon}}. \tag{3.206}$$

Define the $c \times c$ matrices A_i $(i = 0, 1, \cdots, \nu)$ as

$$A_i = H_i H_i', \qquad i = 0, 1, \cdots, \nu. \tag{3.207}$$

Then from (3.203) and (3.204) we have

$$A_i = \overset{s-1}{\underset{\ell=1}{\otimes}} M_{i\ell}, \qquad i = 0, 1, \cdots, \nu \tag{3.208}$$

where

$$M_{i\ell} = \begin{cases} I_{a_\ell} & \text{for } k_\ell \in \psi_i \\ J_{a_\ell} & \text{for } k_\ell \in \psi_i^c \end{cases} \tag{3.209}$$

$(i = 0,1,\cdots,\nu;\ \ell = 1,2,\cdots,s-1)$. Hence

$$E(\underset{\sim}{\bar{y}}) = H\underset{\sim}{g} = \sum_{i=0}^{\nu-p} H_i \underset{\sim}{\beta}_i$$

$$\Sigma = \mathrm{Var}(\underset{\sim}{\bar{y}}) = \sum_{i=\nu-p+1}^{\nu} \sigma_i^2 A_i + \sigma_\epsilon^2 K \tag{3.210}$$

where

$$K = \underset{\tau \in T}{\oplus} n_\tau^{-1}. \tag{3.211}$$

It can be easily verified that $A_iA_j = A_jA_i$ for all $i,j = 0,1,\cdots,\nu$ (see Khuri (1982, p. 2910)). Consequently, there exists an orthogonal matrix Q of order $c \times c$ such that

$$QA_iQ' = \Lambda_i, \quad i = 0,1,\cdots,\nu \tag{3.212}$$

where Λ_i is a diagonal matrix (see Lemma A.5). The construction of Q is described below.

The matrix Q. Recall that $\sum_{i=0}^{\nu} \gamma_{\theta_i(\bar{\theta}_i)}$ in (3.201) is not affected by the imbalance in the last stage of the design. Thus it can be thought of as representing a balanced mixed model with one observation in each τ-cell. Let P_i be the $c \times c$ matrix associated with the sum of squares for the i^{th} effect in the derived model

$$z_\tau = \sum_{i=0}^{\nu} \gamma_{\theta_i}(\bar{\theta}_i). \qquad (3.213)$$

Some properties of $P_i (i = 0, 1, \cdots, \nu)$ are provided in Khuri (1990, p. 181).

Let $\text{rank}(P_i) = m_i$ $(i = 0, 1, \cdots, \nu)$, then

$$\sum_{i=0}^{\nu} m_i = \sum_{i=0}^{\nu} \text{rank}(P_i)$$

$$= \sum_{i=0}^{\nu} \text{tr}(P_i) \qquad (P_i \text{ is idempotent for } i = 0, 1, \cdots, \nu)$$

$$= \text{tr}\Big(\sum_{i=0}^{\nu} P_i\Big)$$

$$= \text{tr}(I_c) \Big(\text{see Khuri (1990) Lemma 3.1(iii)}\Big)$$

$$= c.$$

Let Q_i be a matrix of order $m_i \times c$ whose rows are orthonormal and span the row space of P_i $(i = 0, 1, \cdots, \nu)$. The rows of Q_i can be obtained by orthonormalizing any m_i linearly independent rows (or columns) of P_i $(i = 0, 1, \cdots, \nu)$. The following lemma is proved in Khuri (1990).

Lemma 3.8

The matrices $Q_0, Q_1, \cdots, Q_\nu$ have the following properties:

(a) $Q_0 = \frac{1}{\sqrt{c}} \underset{\sim}{1}'_c$

(b) $Q_i Q'_i = I_{m_i}, \quad i = 0, 1, \cdots, \nu$

$Q_i Q'_j = 0, \quad i \neq j$

(c) $A_j Q'_i = \begin{cases} 0, & \psi_i \not\subset \psi_j, \\ b_j Q'_i, & \psi_i \subset \psi_j, \end{cases} \quad i = 0, 1, \cdots, \nu; \; j = 0, 1, \cdots, \nu,$

where b_j is the number of ones in a typical column of H_j, $j = 0, 1, \cdots, \nu$ (see (3.203)).

A consequence of Lemma 3.8 is that if ψ_i represents a set of subscripts associated with a random effect and ψ_j represents a set of subscripts associated with a fixed effect, then $Q_i A_j = 0$ which implies $Q_i H_j = 0$ $(i = \nu\text{-}p+1, \cdots, \nu; \; j = 0, 1, \cdots, \nu\text{-}p)$. Hence from (3.205), $Q_i H = 0$ $(i = \nu\text{-}p+1, \cdots, \nu)$. The matrix A_j was defined in (3.207).

Write the $c \times c$ matrix Q as

$$Q = [Q'_0 : Q'_1 : \cdots : Q'_\nu]', \tag{3.214}$$

then from Lemma 3.8, Q is orthogonal and simultaneously diagonalizes $A_0, A_1, \cdots, A_\nu$. Furthermore, from (3.210),

$$\mathrm{Var}(Q_i\underset{\sim}{y}) = \sum_{j=\nu-p+1}^{\nu} \sigma_j^2 Q_i A_j Q_i' + \sigma_\epsilon^2 Q_i K Q_i' \tag{3.215}$$

$(i = 0,1,\cdots,\nu)$. However, from Lemma 3.8

$$\begin{aligned}\sum_{j=\nu-p+1}^{\nu} \sigma_j^2 Q_i A_j Q_i' &= \sum_{j\in W_i} b_j \sigma_j^2 I_{m_i} \\ &= \delta_i I_{m_i}, \quad i = 0,1,\cdots,\nu \end{aligned} \tag{3.216}$$

where

$$\begin{aligned} \delta_i &= \sum_{j\in W_i} b_j \sigma_j^2, \quad i = 0,1,\cdots,\nu \\ W_i &= \{j : \nu-p+1 \le j \le \nu, \psi_i \subset \psi_j\}, \quad i = 0,1,\cdots,\nu. \end{aligned} \tag{3.217}$$

Thus from (3.215) we have

$$\mathrm{Var}(Q_i\underset{\sim}{y}) = \delta_i I_{m_i} + \sigma_\epsilon^2 Q_i K Q_i', \quad i = 0,1,\cdots,\nu. \tag{3.218}$$

Also, for $i \neq i'$

$$\begin{aligned}\mathrm{Cov}(Q_i\underset{\sim}{y}, \underset{\sim}{y}'Q_{i'}') &= Q_i \left[\sum_{j=\nu-p+1}^{\nu} \sigma_j^2 A_j + \sigma_\epsilon^2 K\right] Q_{i'}' \\ &= Q_i \left[\sum_{j=\nu-p+1}^{\nu} \sigma_j^2 A_j Q_{i'}' + \sigma_\epsilon^2 K Q_{i'}'\right] \\ &= \sigma_\epsilon^2 Q_i K Q_{i'}'. \end{aligned} \tag{3.219}$$

3.3.3 Analyzing the Variance Components

Introduction. In this section we develop some independent, chi-squared distributed sums of squares that can be used to test hypotheses concerning the variance components. These sums of squares are analogous to the sums of squares associated with the random-effects in an ANOVA table for a balanced mixed model. In fact these sums of squares will reduce to the usual ANOVA sums of squares (associated with the random effects) when the design is balanced.

The key to the construction of these sums of squares is the idea of resampling from the error vector (see Appendix C). It should be noted, however, that resampling is not a "cure-all". Resampling does not guarantee that we will be able to produce an exact test for all the variance components of the model. This should not be too surprising, though, since even in certain balanced models the same phenomenon occurs.

In designs with imbalance affecting the last stage only, resampling, in some sense, counteracts the imbalance by reducing the analysis of the original unbalanced model to that of a balanced one. Consequently we should not expect to do any better (in terms of constructing exact tests) when utilizing resampling in unbalanced models than if the design was balanced to begin with.

Initial development. Recall the definition of Q in (3.214). Let Q_r (r for random) be the $\left(c-\sum_{i=0}^{\nu-p} m_i\right)\times c$ matrix

$$Q_r = [Q'_{\nu-p+1} : Q'_{\nu-p+2} : \cdots : Q'_{\nu}]'. \tag{3.220}$$

Also, define the $\left(c - \sum_{i=0}^{\nu-p} m_i\right) \times 1$ vector $\underset{\sim}{u}_r$ as

$$\underset{\sim}{u}_r = Q_r \bar{\underset{\sim}{y}}. \tag{3.221}$$

The use of the matrix Q_r causes $\underset{\sim}{u}_r$ to have a zero mean vector. In fact, it is easily verified that (recall that $Q_i H = 0$, $i = \nu-p+1, \cdots, \nu$)

$$\begin{aligned} E(\underset{\sim}{u}_r) &= Q_r H \underset{\sim}{g} = \underset{\sim}{0} \\ \mathrm{Var}(\underset{\sim}{u}_r) &= \bigoplus_{i=\nu-p+1}^{\nu} \delta_i I_{m_i} + \sigma_\epsilon^2 G_r \end{aligned} \tag{3.222}$$

where

$$G_r = Q_r K Q'_r. \tag{3.223}$$

Utilizing resampling. The error sum of squares for the original unbalanced model in (3.189) can be expressed as

$$SSE = \sum_{\theta} (y_\theta - \bar{y}_\tau)^2 = \sum_{\tau \in T} \sum_{k_s=1}^{n_\tau} (y_\theta - \bar{y}_\tau)^2 \tag{3.224}$$

where, recall, $\theta = (\tau, k_s)$ and $\bar{y}_\tau$ is defined in (3.198). In matrix form (3.224) becomes

$$SSE = \underset{\sim}{y}' R \underset{\sim}{y} \tag{3.225}$$

where

$$R = I_n - W(W'W)^{-1}W' \tag{3.226}$$

and W is the $n \times c$ matrix

$$W = \bigoplus_{\tau \in T} \underset{\sim}{1}_{n_\tau}. \tag{3.227}$$

In Khuri (1990), the following lemma is proved (see also Lemma B.6).

Lemma 3.9

(a) R is symmetric, idempotent, and has rank n-c.

(b) $DR = 0$, where D is defined in (3.200).

(c) $RX_i = 0$, $i = 0, 1, \cdots, \nu$, where $X_0, X_1, \cdots, X_\nu$ are the matrices in (3.193).

From (3.197) and Lemma 3.9 we can conclude that $\bar{\underset{\sim}{y}}$ and SSE are independent. Furthermore

$$R\Omega/\sigma_\epsilon^2 = R \tag{3.228}$$

thus

$$SSE/\sigma_\epsilon^2 \sim X_{n-c}^2. \tag{3.229}$$

Now, since R is symmetric, idempotent, and has rank n-c, we can express it as

$$R = C_r \Lambda_r C_r' \tag{3.230}$$

where C_r is an orthogonal matrix and Λ_r is a diagonal matrix with n-c ones and c zeros. Without loss of generality, we can partition C_r and Λ_r as

$$\begin{aligned} C_r &= [C_{r1} : C_{r2} : C_{r3}] \\ \Lambda_r &= \mathrm{diag}(I_{\eta_{r1}}, I_{\eta_{r2}}, 0) \end{aligned} \tag{3.231}$$

where

$$\begin{aligned} \eta_{r1} &= c - \sum_{i=0}^{\nu-p} m_i \\ \eta_{r2} &= n-2c + \sum_{i=0}^{\nu-p} m_i \end{aligned} \tag{3.232}$$

and C_{r1}, C_{r2}, and C_{r3} are matrices of orders $n \times \eta_{r1}$, $n \times \eta_{r2}$, and $n \times c$, respectively. Note that $\eta_{r1} + \eta_{r2} = n-c = \mathrm{rank}(R)$. Also $\eta_{r2} > 0$ by Assumption (iv) $\big($see (3.195)$\big)$. Without this assumption, η_{r2} may be nonpositive which would invalidate our methodology.

Since C_r is orthogonal

$$\begin{aligned} C_{ri}' C_{ri} &= I \quad (i = 1,2,3) \\ C_{ri}' C_{rj} &= 0 \quad (i \neq j). \end{aligned} \tag{3.233}$$

Consequently

$$R = C_{r1}C'_{r1} + C_{r2}C'_{r2} \tag{3.234}$$

and

$$\begin{aligned} SSE &= \underset{\sim}{y}'R\underset{\sim}{y} = \underset{\sim}{y}'C_{r1}C'_{r1}\underset{\sim}{y} + \underset{\sim}{y}'C_{r2}C'_{r2}\underset{\sim}{y} \\ &= SSE_{r1} + SSE_{r2} \end{aligned} \tag{3.235}$$

where

$$SSE_{ri} = \underset{\sim}{y}'C_{ri}C'_{ri}\underset{\sim}{y}, \qquad i = 1,2.$$

Clearly $\underset{\sim}{\bar{y}}$ is independent of both SSE_{r1} and SSE_{r2}. Also since $C'_{ri}R = C'_{ri}$ $(i = 1,2)$, it follows that

$$\begin{aligned} C_{ri}C'_{ri}\Omega/\sigma^2_\epsilon &= C_{ri}(C'_{ri}R)\Omega/\sigma^2_\epsilon \\ &= C_{ri}C'_{ri}(R\Omega)/\sigma^2_\epsilon \\ &= C_{ri}C'_{ri}R \qquad \Big(\text{see } (3.228)\Big) \\ &= C_{ri}C'_{ri}, \quad i = 1,2. \end{aligned}$$

Thus $C_{ri}C'_{ri}\Omega/\sigma^2_\epsilon$ is idempotent $(i = 1,2)$ and $C_{r1}C'_{r1}\Omega C_{r2}C'_{r2} = 0$. Hence SSE_{r1} and SSE_{r2} are independent with

$$\mathrm{SSE}_{r1}/\sigma_\epsilon^2 \sim \chi^2_{\eta_{r1}}$$
$$\mathrm{SSE}_{r2}/\sigma_\epsilon^2 \sim \chi^2_{\eta_{r2}}. \tag{3.236}$$

The exact tests. Define the $\eta_{r1} \times 1$ random vector $\underset{\sim}{\omega}_r$ as

$$\underset{\sim}{\omega}_r = \underset{\sim}{u}_r + (\lambda_{r,\max} I_{\eta_{r1}} - G_r)^{\frac{1}{2}} C'_{r1} \underset{\sim}{y} \tag{3.237}$$

where G_r was defined in (3.223); $\lambda_{r,\max}$ is the largest eigenvalue of G_r; $\underset{\sim}{u}_r$ was defined in (3.221); and C'_{r1} was defined in (3.231). The matrix $(\lambda_{r,\max} I_{\eta_{r1}} - G_r)^{\frac{1}{2}}$ is well-defined since $\lambda_{r,\max} I_{\eta_{r1}} - G_r$ is positive semidefinite (p.s.d.).

Partition $\underset{\sim}{u}_r$ in (3.221) as

$$\underset{\sim}{u}_r = (\underset{\sim}{u}'_{\nu\text{-}p+1} : \cdots : \underset{\sim}{u}'_\nu)'$$
$$= (\underset{\sim}{y}' Q'_{\nu\text{-}p+1} : \cdots : \underset{\sim}{y}' Q'_\nu)' \tag{3.238}$$

where $\underset{\sim}{u}_i$ is of dimension m_i; $i = \nu\text{-}p+1, \cdots, \nu$. Partition $\underset{\sim}{\omega}_r$ just like $\underset{\sim}{u}_r$. That is

$$\underset{\sim}{\omega}_r = (\underset{\sim}{\omega}'_{\nu\text{-}p+1} : \cdots : \underset{\sim}{\omega}'_\nu)' \tag{3.239}$$

where $\underset{\sim}{\omega}_i$ is of order $m_i \times 1$, $i = \nu\text{-}p+1, \cdots, \nu$. The major result of this section is now presented.

Theorem 3.7

(a) $E(\underset{\sim}{\omega}_i) = \underset{\sim}{0}$, $i = \nu\text{-}p+1, \cdots, \nu$.

(b) $\underset{\sim}{\omega}_{\nu-p+1}, \cdots, \underset{\sim}{\omega}_\nu$ are independently distributed as normal random vectors with $\underset{\sim}{\omega}_i$ having variance-covariance matrix

$$\mathrm{Var}(\underset{\sim}{\omega}_i) = (\delta_i + \lambda_{r,\max}\sigma_\epsilon^2)I_{m_i}, i = \nu\text{-}p+1, \cdots, \nu. \tag{3.240}$$

(c) $\underset{\sim}{\omega}_r$ is independent of SSE_{r2}.

Proof:

(a) From (3.222), $E(\underset{\sim}{u}_r) = \underset{\sim}{0}$. Also from Lemma 3.9(c), $RX = R[X_0 : \cdots : X_{\nu-p}] = 0$. Thus

$$C'_{r1}X = (C'_{r1}R)X = C'_{r1}(RX) = 0.$$

Consequently the result follows after noting that $E(\underset{\sim}{y}) = X\underset{\sim}{g}$ (see (3.197)) and recalling (3.237).

(b) Clearly $\underset{\sim}{\omega}_r$ is normally distributed. We claim the $\underset{\sim}{u}_r$ is independent of $C'_{r1}\underset{\sim}{y}$. Now, since $D\Omega R = 0$ ($\bar{\underset{\sim}{y}}$ is independent of SSE) it follows that $D\Omega C_{r1} = 0$ $\big(\mathrm{range}(C_{r1}) \subset \mathrm{range}(R)\big)$. Thus $\bar{\underset{\sim}{y}}$ is independent of $C'_{r1}\underset{\sim}{y}$. Therefore $\underset{\sim}{u}_r$ is also independent of $C'_{r1}\underset{\sim}{y}$ as it is a function of $\bar{\underset{\sim}{y}}$ (see (3.221)). Consequently

$$\mathrm{Var}(\underset{\sim}{\omega}_r) = \mathrm{Var}(\underset{\sim}{u}_r) + (\lambda_{r,\max} I_{\eta_{r1}} - G_r)^{\frac{1}{2}} C'_{r1} \Omega C_{r1} (\lambda_{r,\max} I_{\eta_{r1}} - G_r)^{\frac{1}{2}}.$$

But, from (3.228),

$$C'_{r1} \Omega C_{r1} = C'_{r1} R \Omega R C_{r1} = C'_{r1} (\sigma_\epsilon^2 R) C_{r1} = \sigma_\epsilon^2 I_{\eta_{r1}}.$$

Hence, from (3.222),

$$\begin{aligned}\mathrm{Var}(\underset{\sim}{\omega}_r) &= \bigoplus_{i=\nu\text{-}p+1}^{\nu} \delta_i I_{m_i} + \sigma_\epsilon^2 G_r + (\lambda_{r,\max} I_{\eta_{r1}} - G_r)\sigma_\epsilon^2 \\ &= \bigoplus_{i=\nu\text{-}p+1}^{\nu} \delta_i I_{m_i} + \lambda_{r,\max}\sigma_\epsilon^2 I_{\eta_{r1}}.\end{aligned}$$

Consequently $\underset{\sim}{\omega}_{\nu\text{-}p+1}, \cdots, \underset{\sim}{\omega}_\nu$ are independent and

$$\mathrm{Var}(\underset{\sim}{\omega}_i) = (\delta_i + \lambda_{r,\max}\sigma_\epsilon^2) I_{m_i}, \quad i = \nu\text{-}p+1, \cdots, \nu.$$

(c) Since $\underset{\sim}{u}_r$ is independent of SSE, it follows that it is independent of SSE_{r2} $\big(\mathrm{range}(C_{r2}) \subset \mathrm{range}(R)\big)$. Also, since SSE_{r1} and SSE_{r2} are independent, it follows that $C'_{r1}\underset{\sim}{y}$ is independent of SSE_{r2}. Thus $\underset{\sim}{\omega}_r$ and SSE_{r2} are independent. □

Let us define SS_{ri} as

$$\mathrm{SS}_{ri} = \underset{\sim}{\omega}'_i \underset{\sim}{\omega}_i, \quad i = \nu\text{-}p+1, \cdots, \nu. \tag{3.241}$$

Then, from Theorem 3.7,

(i) $SS_{r,\nu-p+1},\cdots,SS_{r\nu}$, and SSE_{r2} are mutually independent.

(ii) $SS_{ri}/(\delta_i+\lambda_{r,\max}\sigma_\epsilon^2)\sim\chi^2_{m_i}$, $i=\nu-p+1,\cdots,\nu$.

Thus, analyses concerning the variance components $\sigma^2_{\nu-p+1},\cdots,\sigma^2_\nu$, can proceed using $SS_{r,\nu-p+1},\cdots,SS_{r\nu}$, and SSE_{r2}. To illustrate the idea let us suppose we wish to test $H_0:\sigma_i^2=0$ vs $H_a:\sigma_i^2\neq 0$, $i=\nu-p+1,\cdots,\nu$. Also, suppose that there exists a $j(\neq i)\in\{\nu-p+1,\cdots,\nu+1\}$ such that under $H_0:\sigma_i^2=0$

$$E\left[\frac{SS_{ri}}{m_i}\right]=\delta_j+\lambda_{r,\max}\sigma_\epsilon^2. \tag{3.242}$$

(For notational convenience take $SS_{r,\nu+1}\equiv SSE_{r2}$, $\delta_{\nu+1}\equiv 0$, and $m_{\nu+1}\equiv\eta_{r2}/\lambda_{r,\max}$.) Then a test statistic for this hypothesis is

$$F_{ri}=\frac{SS_{ri}/m_i}{SS_{rj}/m_j}.$$

Under H_0, $F_{ri}\sim F_{m_i,m_j}$. From (3.217) we would reject H_0 for large values of F_{ri}.

If no such j exists satisfying (3.242) then no exact test can be developed based on our methodology. In such a situation we must resort to Satterthwaite's procedure to construct an approximate test for testing this hypothesis.

3.3.4 Analyzing the Fixed-Effects Parameters

Introduction. In this section we concentrate on the fixed portion of the model (3.189). When possible, exact tests will be developed to test hypotheses about the unknown parameters in the model. When no such exact test can be developed we will utilize Satterthwaite's procedure (after making an appropriate transformation) to develop an approximate test.

Initial development. To begin with, let $\underset{\sim}{u}$ be the $c \times 1$ random vector defined as

$$\underset{\sim}{u} = Q\underset{\sim}{\bar{y}} \tag{3.243}$$

where Q is the $c \times c$ orthogonal matrix defined in (3.214). If we let Q_f (f for fixed) be the $\sum_{i=0}^{\nu-p} m_i \times c$ matrix

$$Q_f = (Q_0' : Q_1' : \cdots : Q_{\nu-p}')', \tag{3.244}$$

then, recalling (3.220), we can write (3.243) as

$$\underset{\sim}{u} = \begin{bmatrix} Q_f\underset{\sim}{\bar{y}} \\ Q_r\underset{\sim}{\bar{y}} \end{bmatrix} \overset{\text{say}}{=} \begin{bmatrix} \underset{\sim}{u}_f \\ \underset{\sim}{u}_r \end{bmatrix} \tag{3.245}$$

where $\underset{\sim}{u}_f' = (\underset{\sim}{u}_0' : \underset{\sim}{u}_1' : \cdots : \underset{\sim}{u}_{\nu-p}')'$. It is easily verified that

$$E(\underset{\sim}{u}) = \begin{bmatrix} Q_f H \underset{\sim}{g} \\ \underset{\sim}{0} \end{bmatrix}$$

$$\mathrm{Var}(\underset{\sim}{u}) = \bigoplus_{i=0}^{\nu} \delta_i I_{m_i} + \sigma_\epsilon^2 G \tag{3.246}$$

where

$$G = QKQ' \tag{3.247}$$

and H and K are the matrices defined in (3.205) and (3.211), respectively. Formula (3.246) follows from (3.215)-(3.219) and the fact that $Q_iH = 0$, $i = \nu\text{-}p+1, \cdots, \nu$ (see Lemma 3.8(c)). Now, from (3.246) it follows that

$$\mathrm{Var}(\underset{\sim}{u}_f) = \bigoplus_{i=0}^{\nu-p} \delta_i I_{m_i} + \sigma_\epsilon^2 G_f \tag{3.248}$$

where

$$G_f = Q_f K Q_f'. \tag{3.249}$$

Also,

$$\mathrm{Cov}(\underset{\sim}{u}_f, \underset{\sim}{u}_r') = \sigma_\epsilon^2 Q_f K Q_r'. \tag{3.250}$$

Thus $\underset{\sim}{u}_f$ and $\underset{\sim}{u}_r$ are not independent. (They would be if the design was balanced (see Lemma 3.8(b))).

Now, recall that $m_i = \mathrm{rank}(P_i) = \mathrm{rank}(Q_i)$, $i = 0, 1, \cdots, \nu$ (see the discussion after (3.213)). Also, let $\underset{\sim}{z}$ be the $c \times 1$ vector consisting of the c z_τ's from the derived model (3.213). Then, from Khuri's (1990) Lemma 3.1 (iii), we can express the total sum of squares associated with model (3.213) as

$$\underset{\sim}{z}'\underset{\sim}{z} = \underset{\sim}{z}'\Big(\sum_{i=0}^{\nu} P_i\Big)\underset{\sim}{z} = \sum_{i=0}^{\nu} \underset{\sim}{z}'P_i\underset{\sim}{z}. \tag{3.251}$$

Since Q is orthogonal, we can equivalently express (3.251) as $\Big($see (3.214)$\Big)$

$$\underset{\sim}{z}'\underset{\sim}{z} = \underset{\sim}{z}'Q'Q\underset{\sim}{z} = \underset{\sim}{z}'\Big(\sum_{i=0}^{\nu} Q_i'Q_i\Big)\underset{\sim}{z} = \sum_{i=0}^{\nu} \underset{\sim}{z}'Q_i'Q_i\underset{\sim}{z}.$$

Consequently, since $Q_i'Q_i$ is idempotent and range(P_i) = range(Q_i'),

$$P_i = Q_i'Q_i, \quad i = 0,1,\cdots,\nu. \tag{3.252}$$

Lemma 3.10

(a) $\text{rank}(Q_iH) = m_i$, $i = 0,1,\cdots,\nu\text{-}p$,

(b) $\text{rank}(H) = \sum_{i=0}^{\nu-p} m_i$.

(c) If $\underset{\sim}{\tau}_i = Q_i'Q_iH\underset{\sim}{g}$, $i = 0,1,\cdots,\nu\text{-}p$, then $\underset{\sim}{\tau}_0, \underset{\sim}{\tau}_1, \cdots, \underset{\sim}{\tau}_{\nu-p}$ are linearly independent and span the space of all linearly estimable functions of $\underset{\sim}{g}$, the vector of fixed effects in model (3.196).

Proof:

(a) By Lemma 3.8, (3.205), and (3.207)

$$Q_iHH'Q_i' = Q_i\Big(\sum_{j=0}^{\nu-p} A_j\Big)Q_i'$$

$$= \Big(\sum_{j \in U_i} b_j\Big) I_{m_i}$$

$$= \delta_i^* I_{m_i}, \quad i = 0, 1, \cdots, \nu\text{-}p$$

where

$$U_i = \{j : 0 \le j \le \nu\text{-}p,\ \psi_i \subset \psi_j\}, \quad i = 0, 1, \cdots, \nu\text{-}p$$

$$\delta_i^* = \sum_{j \in U_i} b_j .$$

Thus

$$\operatorname{rank}(Q_i H) = \operatorname{rank}(Q_i H H' Q_i')$$

$$= \operatorname{rank}(\delta_i^* I_{m_i}) = m_i, \quad i = 0, 1, \cdots, \nu\text{-}p.$$

(b) First note that

$$Q_f H H' Q_f' = \begin{bmatrix} Q_0 \\ \vdots \\ Q_{\nu\text{-}p} \end{bmatrix} \Big(\sum_{j=0}^{\nu\text{-}p} A_j\Big) \left[Q_0' : \cdots : Q_{\nu\text{-}p}'\right].$$

From Lemma 3.8 we have

$$Q_i\Big(\sum_{j=0}^{\nu\text{-}p} A_j\Big) Q_{i'}' = \begin{cases} \delta_i^* I_{m_i}, & i = i' = 0, 1, \cdots, \nu\text{-}p \\ 0, & i \ne i'. \end{cases}$$

Consequently,

$$Q_f HH'Q_f' = \overset{\nu-p}{\underset{i=0}{\oplus}} \delta_i^* I_{m_i} .$$

Thus

$$\text{rank}(H) \geq \text{rank}(Q_f H) = \text{rank}(Q_f HH'Q_f') = \sum_{i=0}^{\nu-p} m_i . \qquad (3.253)$$

Conversely, from Lemma 3.8 and the fact that Q is orthogonal, we have

$$\begin{aligned} H = Q'QH &= \Big(\sum_{i=0}^{\nu} Q_i'Q_i \Big) H \\ &= \Big(\sum_{i=0}^{\nu-p} Q_i'Q_i \Big) H + \Big(\sum_{i=\nu-p+1}^{\nu} Q_i'Q_i \Big) H \\ &= \Big(\sum_{i=0}^{\nu-p} Q_i'Q_i \Big) H . \end{aligned}$$

Since

$$\begin{aligned} \Big(\sum_{i=0}^{\nu-p} Q_i'Q_i \Big) \Big(\sum_{j=0}^{\nu-p} Q_j'Q_j \Big) &= \sum_{i=0}^{\nu-p} Q_i'Q_i + \sum_{i \neq j} Q_i'Q_iQ_j'Q_j \\ &= \sum_{i=0}^{\nu-p} Q_i'Q_i , \end{aligned}$$

we see that $\sum_{i=0}^{\nu-p} Q_i'Q_i$ is idempotent and thus $\text{rank}\Big(\sum_{i=0}^{\nu-p} Q_i'Q_i \Big)$ $= \text{tr}\Big(\sum_{i=0}^{\nu-p} Q_i'Q_i \Big) = \sum_{i=0}^{\nu-p} \text{tr}(Q_i'Q_i) = \sum_{i=0}^{\nu-p} m_i .$ Hence

$$\mathrm{rank}(H) = \mathrm{rank}\left[\left(\sum_{i=0}^{\nu-p} Q_i' Q_i\right) H\right]$$

$$\leq \mathrm{rank}\left(\sum_{i=0}^{\nu-p} Q_i' Q_i\right) = \sum_{i=0}^{\nu-p} m_i . \tag{3.254}$$

Therefore, (3.253) and (3.254) imply that $r(H) = \sum_{i=0}^{\nu-p} m_i$.

(c) Recall that $\underset{\sim}{\tau}_i = Q_i' Q_i H \underset{\sim}{g}$, $i = 0, 1, \cdots, \nu-p$. Suppose that $\underset{\sim}{\tau}_0$, $\underset{\sim}{\tau}_1, \cdots, \underset{\sim}{\tau}_{\nu-p}$ were linearly dependent, then there would exist constants $a_0^*, a_1^*, \cdots, a_{\nu-p}^*$, not all zero, such that

$$\sum_{i=0}^{\nu-p} a_i^* Q_i' Q_i H \underset{\sim}{g} = \underset{\sim}{0} .$$

This implies that $\sum_{i=0}^{\nu-p} a_i^* Q_i' Q_i H = 0$. Premultiplying by $Q_j (0 \leq j \leq \nu-p)$ results in

$$Q_j\left(\sum_{i=0}^{\nu-p} a_i^* Q_i' Q_i H\right) = a_j^* Q_j H = 0 .$$

Hence we conclude that $a_j^* = 0$. Consequently $a_i^* = 0$ for $i = 0, 1, \cdots, \nu-p$ which is a contradiction. Therefore $\underset{\sim}{\tau}_0, \underset{\sim}{\tau}_1, \cdots, \underset{\sim}{\tau}_{\nu-p}$ must be linearly independent.

Now,

$$H\underset{\sim}{g} = Q'QH\underset{\sim}{g} = \left(\sum_{i=0}^{\nu} Q_i' Q_i\right) H\underset{\sim}{g}$$

$$= \sum_{i=0}^{\nu-p} Q_i' Q_i H \underset{\sim}{g} = \sum_{i=0}^{\nu-p} \underset{\sim}{\tau}_i .$$

Thus, any estimable linear function, $\underset{\sim}{\ell}'H\underset{\sim}{g}$, of $\underset{\sim}{g}$ can be written as a linear combination of the elements of $\underset{\sim}{\tau}_0,\underset{\sim}{\tau}_1,\cdots,\underset{\sim}{\tau}_{\nu-p}$ of the form $\sum_{i=0}^{\nu-p}\underset{\sim}{\ell}'\underset{\sim}{\tau}_i$. Therefore, $\underset{\sim}{\tau}_0,\underset{\sim}{\tau}_1,\cdots,\underset{\sim}{\tau}_{\nu-p}$ span the space of all linear estimable functions of $\underset{\sim}{g}$. □

From Lemma 3.10 we can restrict our attention to developing methodology concerning $\underset{\sim}{\tau}_0,\underset{\sim}{\tau}_1,\cdots,\underset{\sim}{\tau}_{\nu-p}$. Before we proceed, let us recall that (see (3.246))

$$\begin{aligned} \underset{\sim}{u}_i &\sim N(Q_iH\underset{\sim}{g},\ \delta_iI_{m_i}+\sigma_\epsilon^2Q_iKQ_i'),\ i=0,1,\cdots,\nu\text{-}p \\ \underset{\sim}{u}_{i_0} &\sim N(\underset{\sim}{0},\delta_{i_0}I_{m_{i_0}}+\sigma_\epsilon^2Q_{i_0}KQ_{i_0}'),\ i_0=\nu\text{-}p\text{+}1,\cdots,\nu. \end{aligned} \tag{3.255}$$

In (3.255) notice that

$$Q_iH\underset{\sim}{g}=Q_i\underset{\sim}{\tau}_i,\ i=0,1,\cdots,\nu\text{-}p. \tag{3.256}$$

The vector of "proper error terms". If the design was balanced and we wished to test a testable hypothesis concerning an estimable linear function of the fixed effects in the model, we would do so by scanning the "E(MS)" column of the corresponding ANOVA table for a mean square which could serve as an "error" term (if one existed) for the test statistic used in testing our hypothesis. This mean square would be chosen in such a way that under the null hypothesis our test statistic would have an F-distribution. If no such mean square existed then, typically, we would utilize Satterthwaite's procedure to develop an approximate test.

When the design is unbalanced there generally does not exist a mean square that can serve as a proper error term. Consequently exact tests (based on the ANOVA sums of squares) do not usually exist for the above mentioned hypothesis testing problem. The most common approximate test is based upon Satterthwaite's procedure. Unfortunately the sums of squares involved in Satterthwaite's quasi F-ratio are typically neither independent nor chi-squared distributed. Either of these "failures" is enough to invalidate Satterthwaite's methodology. This has led some authors to advocate the use of traditional (balanced) methods even in the presence of imbalance $\big($Bliss (1968) or Tietjen (1977), for example$\big)$.

We now present a technique that will allow us to construct exact tests (when such exist) for estimable linear functions of the fixed effects in model (3.196). This will be done by using resampling to make a transformation that will result in a set of mutually independent, (scaled) chi-squared distributed sums of squares. The idea will be illustrated by considering a three-way cross-classification design in which factors A and B are fixed while factor C is random.

Let the random vector $\underset{\sim}{v}$ be defined as

$$\underset{\sim}{v} = \begin{bmatrix} \underset{\sim}{v}_f \\ \underset{\sim}{v}_r \end{bmatrix} \qquad (3.257)$$

where $\underset{\sim}{v}_f$ and $\underset{\sim}{v}_r$ are determined as follows:

(i) Put $\underset{\sim}{u}_i$ $(i = 0, 1, \cdots, \nu\text{-}p)$ in $\underset{\sim}{v}_f$ if and only if there exists an $i_0 \in \{\nu\text{-}p\text{+}1, \cdots, \nu\}$ for which $\delta_i = \delta_{i_0}$.

(ii) Put the corresponding $\underset{\sim}{u}_{i_0}$ $(i_0 = \nu\text{-}p+1, \cdots, \nu)$ in $\underset{\sim}{v}_r$.

Thus, $\underset{\sim}{v}_f$ consists of those vectors for which a proper error term exists and $\underset{\sim}{v}_r$ consists of the corresponding vectors used in those proper error terms. Recall that $\underset{\sim}{u}_k = Q_k\underset{\sim}{\bar{y}}$ where Q_k is defined in (3.214), $k = 0,1,\cdots,\nu$. Also δ_k $(k = 0,1,\cdots,\nu)$ is defined in (3.217). It is also convenient to introduce the following notation:

$$\mathcal{I}_f = \{0 \le i \le \nu\text{-}p: \ \underset{\sim}{u}_i \text{ is part of } \underset{\sim}{v}_f\} \tag{3.258}$$

$$\mathcal{I}_r = \{\nu\text{-}p+1 \le i_0 \le \nu: \ \underset{\sim}{u}_{i_0} \text{ is part of } \underset{\sim}{v}_r\} \tag{3.259}$$

$$\mathcal{I} \equiv \mathcal{I}_f \cup \mathcal{I}_r .$$

We note that $\mathcal{I}_f$ and $\mathcal{I}_r$ (and hence $\mathcal{I}$) may be empty. In such cases we cannot apply our procedure.

Example 3.2

Consider a three-way cross-classification design in which factors A and B are fixed while factor C is random. The model can be expressed as

$$y_{ijk\ell} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \gamma_k + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \epsilon_{ijk\ell} \tag{3.260}$$

$(i = 1,2,\cdots,a;\ j = 1,2,\cdots,b;\ k = 1,2,\cdots,c;\ \ell = 1,2,\cdots,n_{ijk})$.

Following the notation in Section 3.3.2: $\nu = 7$, $p = 4$, and γ_k, $(\alpha\gamma)_{ik}$, $(\beta\gamma)_{jk}$, $(\alpha\beta\gamma)_{ijk}$, and $\epsilon_{ijk\ell}$ are independent, normally distributed random variables with zero means and variances σ_4^2, σ_5^2, σ_6^2, σ_7^2, and σ_ϵ^2, respectively. Table 3.3 summarizes the pertinent information needed in the determination of $\underset{\sim}{v}$.

From Table 3.3 we see that

$$\begin{aligned} \underset{\sim}{v}_f &= [\underset{\sim}{u}_0' : \underset{\sim}{u}_1' : \underset{\sim}{u}_2' : \underset{\sim}{u}_3']' \\ \underset{\sim}{v}_r &= [\underset{\sim}{u}_4' : \underset{\sim}{u}_5' : \underset{\sim}{u}_6' : \underset{\sim}{u}_7']'. \end{aligned} \tag{3.261}$$

Thus, in this example, the vector $\underset{\sim}{v}$ is exactly the vector $\underset{\sim}{u}$. It is interesting to note that if we had instead assumed factor A fixed and factors B and C random then $\underset{\sim}{v}$ would not consist of any elements and our procedure could not be applied. That is, no exact tests (based on our methodology) could be constructed for any estimable linear function of the fixed effects. We would thus need to apply Satterthwaite's procedure. More is said about this in the sequel.

Utilizing resampling. Recall (3.225) and (3.226). Instead of the partitioning of R as described in (3.230)-(3.232), we now express R as

$$R = C_f \Lambda_f C_f' \tag{3.262}$$

where C_f is an orthogonal matrix and Λ_f is a diagonal matrix consisting of n-c ones and c zeros. We can partition C_f and Λ_f as

Table 3.3 Information needed to determine the vector $\underset{\sim}{v}$ in (3.257) for a three-way cross-classification model in which Factors A and B are fixed while Factor C is random

i	ψ_i	b_i	W_i (see (3.217))	δ_i (see (3.217))
0	ϕ	abc	{4,5,6,7}	$ab\sigma_4^2+b\sigma_5^2+a\sigma_6^2+\sigma_7^2$
1	{i}	bc	{5,7}	$b\sigma_5^2+\sigma_7^2$
2	{j}	ac	{6,7}	$a\sigma_6^2+\sigma_7^2$
3	{i,j}	c	{7}	σ_7^2
4	{k}	ab	{4,5,6,7}	$ab\sigma_4^2+b\sigma_5^2+a\sigma_6^2+\sigma_7^2$
5	{i,k}	b	{5,7}	$b\sigma_5^2+\sigma_7^2$
6	{j,k}	a	{6,7}	$a\sigma_6^2+\sigma_7^2$
7	{i,j,k}	1	{7}	σ_7^2

$$C_f=[C_{f1}:C_{f2}:C_{f3}]$$

$$\Lambda_f=\mathrm{diag}(I_{\eta_{f1}},I_{\eta_{f2}},0), \tag{3.263}$$

where

$$\eta_{f1}=c-\sum_{i\in\mathcal{I}^c}m_i$$

$$\eta_{f2}=n-2c+\sum_{i\in\mathcal{I}^c}m_i \tag{3.264}$$

and $\mathcal{I}^c$ is the complement of $\mathcal{I}$ with respect to $\{0,1,\cdots,\nu\}$. If $\mathcal{I}^c=\phi$ then $\eta_{f1}=c$ and $\eta_{f2}=n-2c$. Note that $\eta_{f1}+\eta_{f2}=n-c=\mathrm{rank}(R)$. Also, by (3.195), $\eta_{f2}>0$.

As $\underset{\sim}{v}$ $\Big($in (3.257)$\Big)$ consists of those $\underset{\sim}{u}_i$ for which $i \in \mathscr{I}$, let Q_v consist of all those Q_i for which $i \in \mathscr{I}$. In Example 3.2 $Q_v \equiv Q = [Q_0' : Q_1' : \cdots : Q_\nu']'$. If $\underset{\sim}{v}$ is void of elements (or, equivalently $\mathscr{I} = \phi$) then Q_v will not contain any elements and, as mentioned previously, our procedure cannot be applied.

Now, define the random vector $\underset{\sim}{\omega}_f$ as

$$\underset{\sim}{\omega}_f = \underset{\sim}{v} + (\lambda_{f,\max} I_{\eta_{f1}} - G_f)^{\frac{1}{2}} C_{f1}' \underset{\sim}{y} \tag{3.265}$$

where $\lambda_{f,\max}$ is the largest eigenvalue of G_f and G_f is the matrix

$$G_f = Q_v K Q_v'. \tag{3.266}$$

The matrix $(\lambda_{f,\max} I_{\eta_{f1}} - G_f)^{\frac{1}{2}}$ is well-defined since $(\lambda_{f,\max} I_{\eta_{f1}} - G_f)$ is p.s.d.

Before we proceed we need the following notation: Suppose $S = \{\rho_0, \rho_1, \cdots, \rho_t\}$ where each ρ_i is an integer satisfying $\rho_0 < \rho_1 < \cdots < \rho_t$ and $t+1$ is the number of elements in S. Then

$$\underset{\sim}{w} = (\underset{\sim}{w}_j : j \in S) \equiv (\underset{\sim}{w}_{\rho_0}' : \underset{\sim}{w}_{\rho_1}' : \cdots : \underset{\sim}{w}_{\rho_t}')'.$$

Now, partition $\underset{\sim}{v}$ as

$$\underset{\sim}{v} = (\underset{\sim}{v}_i : i \in \mathscr{I}), \tag{3.267}$$

where each $\underset{\sim}{v}_i$ is of dimension m_i, $i \in \mathscr{I}$. We note that $\{\underset{\sim}{v}_i : i \in \mathscr{I}\}$ represents a subset of $\{\underset{\sim}{u}_0, \underset{\sim}{u}_1, \cdots, \underset{\sim}{u}_\nu\}$. Partition $\underset{\sim}{\omega}_f$ just like $\underset{\sim}{v}$.

That is

$$\underset{\sim}{\omega}_f = (\underset{\sim}{\omega}_{fi} : i \in \mathcal{I}) \tag{3.268}$$

where $\underset{\sim}{\omega}_{fi}$ is of order $m_i \times 1$, $i \in \mathcal{I}$. The key result of this section is given in the next theorem which provides some distributional properties for $\{\underset{\sim}{\omega}_{fi} : i \in \mathcal{I}\}$ and justifies the transformation made in (3.265).

Theorem 3.8

(a) $$E(\underset{\sim}{\omega}_{fi}) = \begin{cases} Q_i \underset{\sim}{\tau}_i, & i \in \mathcal{I}_f \\ \underset{\sim}{0}, & i \in \mathcal{I}_r \end{cases} \tag{3.269}$$

where $\underset{\sim}{\tau}_i$ $(i = 0, 1, \cdots, \nu\text{-}p)$ is defined in Lemma 3.10(c).

(b) $\{\underset{\sim}{\omega}_{fi} : i \in \mathcal{I}\}$ consist of independently distributed, normal random vectors with $\underset{\sim}{\omega}_{fi}$ having variance-covariance matrix

$$\mathrm{Var}(\underset{\sim}{\omega}_{fi}) = (\delta_i + \lambda_{f,\max}\sigma_\epsilon^2) I_{m_i}, \quad i \in \mathcal{I}. \tag{3.270}$$

Proof:

(a) From Lemma 3.9(c), $RX_i = 0$, $i = 0, 1, \cdots, \nu$. Therefore, since $\mathrm{range}(C_{f1}) \subset \mathrm{range}(R)$ and R is symmetric, it follows that $C'_{f1} X_i = 0$, $i = 0, 1, \cdots, \nu$. Consequently

$$E(C'_{f1} \underset{\sim}{y}) = C'_{f1} X \underset{\sim}{g} = \underset{\sim}{0}.$$

(recall that $X = [X_0 : X_1 : \cdots : X_{\nu-p}]$). Since

$$E(\underset{\sim}{u}_i) = \begin{cases} Q_i \underset{\sim}{\tau}_i, & i = 0, 1, \cdots, \nu-p \\ \underset{\sim}{0}, & i = \nu-p+1, \cdots, \nu \end{cases}$$

(see (3.255)), it must be that

$$E(\underset{\sim}{v}_i) = \begin{cases} Q_i \underset{\sim}{\tau}_i, & i \in \mathcal{I}_f \\ \underset{\sim}{0}, & i \in \mathcal{I}_r. \end{cases}$$

Thus the result follows after recalling the definition of $\underset{\sim}{\omega}_f$ given in (3.265).

(b) Clearly $\underset{\sim}{\omega}_{fi}$ is normally distributed for each $i \in \mathcal{I}$. Now recall that $\underset{\sim}{u}_i = Q_i \underset{\sim}{\bar{y}}$, $i = 0, 1, \cdots, \nu$, (see (3.243)). Since $\underset{\sim}{v}$ consist of exactly those $\underset{\sim}{u}_i$ for which $i \in \mathcal{I}$ and Q_v consists of the corresponding Q_i, it follows that

$$\underset{\sim}{v} = Q_v \underset{\sim}{\bar{y}}. \tag{3.271}$$

Since $\underset{\sim}{\bar{y}}$ is independent of SSE, $D\Omega R = 0$. Since range$(C_{f1}) \subset$ range(R), $D\Omega C_{f1} = 0$. Thus $\underset{\sim}{\bar{y}}$ is independent of $C'_{f1}\underset{\sim}{y}$. Consequently $\underset{\sim}{v}$ is also independent of $C'_{f1}\underset{\sim}{y}$ as it is a function of $\underset{\sim}{\bar{y}}$. Therefore

$$\mathrm{Var}(\underset{\sim}{\omega}_f) = \mathrm{Var}(\underset{\sim}{v}) + (\lambda_{f,\max} I_{\eta_{f1}} - G_f)^{\frac{1}{2}} C'_{f1} \Omega C_{f1} (\lambda_{f,\max} I_{\eta_{f1}} - G_f)^{\frac{1}{2}}.$$

By the now familiar argument, it is easy to show that

$$C'_{f1}\Omega C_{f1} = \sigma_\epsilon^2 I_{\eta_{f1}}.$$

Also, from (3.217) and (3.218),

$$\mathrm{Var}(\underset{\sim}{y}) = \underset{i \in \mathcal{I}}{\oplus} \delta_i I_{m_i} + \sigma_\epsilon^2 G_f.$$

Hence, after some simplification, we obtain

$$\mathrm{Var}(\underset{\sim}{\omega}_f) = \underset{i \in \mathcal{I}}{\oplus} (\delta_i + \sigma_\epsilon^2 \lambda_{f,\max}) I_{m_i}.$$

Thus

$$\mathrm{Var}(\underset{\sim}{\omega}_{fi}) = (\delta_i + \sigma_\epsilon^2 \lambda_{f,\max}) I_{m_i}, \quad i \in \mathcal{I}. \qquad \square$$

From Theorem 3.8, we can conclude:

$$\underset{\sim}{\omega}_{fi} \sim N(Q_i \underset{\sim}{\tau}_i,\ \xi_{fi} I_{m_i}) \text{ for } i \in \mathcal{I}_f$$

and

$$\underset{\sim}{\omega}_{f_{i_0}} \sim N(\underset{\sim}{0},\ \xi_{f_{i_0}} I_{m_{i_0}}) \text{ for } i_0 \in \mathcal{I}_r \qquad (3.272)$$

where

$$\xi_{fk} = \delta_k + \sigma_\epsilon^2 \lambda_{f,\max}, \quad k \in \mathcal{I}. \qquad (3.273)$$

The exact procedures. First recall, from Lemma 3.10(c), that $\underset{\sim}{\tau}_i = Q'_i Q_i H \underset{\sim}{g}$, $i = 0, 1, \cdots, \nu\text{-}p$. Consequently,

$$Q_i'Q_i\underset{\sim}{\tau}_i=\underset{\sim}{\tau}_i,\quad i=0,1,\cdots,\nu\text{-}p. \tag{3.274}$$

Thus, constructing exact simultaneous confidence intervals for all linear combinations, $\underset{\sim}{d}'\underset{\sim}{\tau}_i$, of $\underset{\sim}{\tau}_i$ ($\underset{\sim}{d}\in\mathbb{R}^c$) is equivalent to constructing exact simultaneous confidence intervals for all $\underset{\sim}{r}'Q_i\underset{\sim}{\tau}_i$, where $\underset{\sim}{r}'=\underset{\sim}{d}'Q_i'$ and hence is in the row space of Q_i', $i=0,1,\cdots,\nu\text{-}p$.

From (3.272) we can conclude that

$$\begin{aligned}&\text{(i)}\quad (\underset{\sim}{\omega}_{fi}-Q_i\underset{\sim}{\tau}_i)'(\underset{\sim}{\omega}_{fi}-Q_i\underset{\sim}{\tau}_i)/\xi_{fi}\sim\chi^2_{m_i},\ i\in\mathcal{I}_f\\ &\text{(ii)}\quad \underset{\sim}{\omega}_{fi_0}'\underset{\sim}{\omega}_{fi_0}/\xi_{fi_0}\sim\chi^2_{m_{i_0}},\ i_0\in\mathcal{I}_r.\end{aligned} \tag{3.275}$$

By our construction of $\underset{\sim}{v}$ in (3.257), for each $i\in\mathcal{I}_f$ there exists an $i_0\in\mathcal{I}_r$ such that $\delta_{i_0}=\delta_i$ (implying $\xi_{fi_0}=\xi_{fi}$). Therefore if $i\in\mathcal{I}_f$ and $i_0\in\mathcal{I}_r$ are such that $\xi_{fi_0}=\xi_{fi}$ then, for $MS_{fi_0}=\underset{\sim}{\omega}_{fi_0}'\underset{\sim}{\omega}_{fi_0}/m_{i_0}$,

$$E\left[MS_{fi_0}\right]=\xi_{fi_0}=\xi_{fi} \tag{3.276}$$

and

$$\frac{(\underset{\sim}{\omega}_{fi}-Q_i\underset{\sim}{\tau}_i)'(\underset{\sim}{\omega}_{fi}-Q_i\underset{\sim}{\tau}_i)/m_i}{MS_{fi_0}}\sim F_{m_i,m_{i_0}}. \tag{3.277}$$

Hence, simultaneous confidence intervals on all linear functions, $\underset{\sim}{r}'Q_i\underset{\sim}{\tau}_i$, of $Q_i\underset{\sim}{\tau}_i$ $\left(\underset{\sim}{r}\in\text{range}(Q_i)\right)$ can be obtained from the formula

$$P\left[\underset{\sim}{r}'Q_i\underset{\sim}{\tau}_i \in \underset{\sim}{r}'\underset{\sim}{\omega}_{fi} \pm \left(m_i MS_{fi_0} F_{\alpha,m_i,m_{i_0}} \underset{\sim}{r}'\underset{\sim}{r}\right)^{\frac{1}{2}}\right] = 1-\alpha \qquad (3.278)$$

$(i \in \mathcal{S}_f,\ i_0 \in \mathcal{S}_r$ such that $\xi_{fi_0} = \xi_{fi})$. We note that $\underset{\sim}{r}'\underset{\sim}{\omega}_{fi} = \underset{\sim}{d}'Q_i'\underset{\sim}{\omega}_{fi}$ and thus since $Q_i'Q_i$ is idempotent, $\underset{\sim}{d}'Q_i'\underset{\sim}{\omega}_{fi}$ is the least squares estimator of $\underset{\sim}{d}'\underset{\sim}{\tau}_i$ $i \in \mathcal{S}_f$ (see (3.272)). Formula (3.278) is based on Scheffé's S-method (Sheffé (1959, pp. 68-72)).

To test $H_0: L'\underset{\sim}{\tau}_i = \underset{\sim}{\ell}$ for $i \in \mathcal{S}_f$, where L' has full row rank r_L, we proceed as follows: Recall (3.272), let $\hat{\underset{\sim}{\tau}}_i = Q_i'\underset{\sim}{\omega}_{fi}$ be the least squares estimator of $\underset{\sim}{\tau}_i$, $i \in \mathcal{S}_f$. then

$$L'\hat{\underset{\sim}{\tau}}_i - \underset{\sim}{\ell} = L'Q_i'\underset{\sim}{\omega}_{fi} - \underset{\sim}{\ell} \sim N(L'\underset{\sim}{\tau}_i - \underset{\sim}{\ell},\ \xi_{fi}L'Q_i'Q_iL) \qquad (3.279)$$

$(i \in \mathcal{S}_f)$. Because we are assuming that the above hypothesis is testable for all $i \in \mathcal{S}_f$, $L'\underset{\sim}{\tau}_i$ consists of a set of linearly estimable functions of $\underset{\sim}{\tau}_i$, $i \in \mathcal{S}_f$. Thus, there exists a full row rank matrix M $(\text{rank}(M) = r_L)$ such that

$$L' = M'Q_i'Q_i, \quad i \in \mathcal{S}_f. \qquad (3.280)$$

Then, taking $r_L \le m_i$, $i \in \mathcal{S}_f$, it is easily shown that $M'Q_i'$ has full row rank r_L, $i \in \mathcal{S}_f$. $\big(r_L = \text{rank}(L') \le \text{rank}(M'Q_i')$, $i \in \mathcal{S}_f$; but $M'Q_i'$ is $r_L \times m_i$ and $r_L \le m_i$ and so $\text{rank}(M'Q_i') \le r_L$, $i \in \mathcal{S}_f\big)$. Furthermore

$$L'Q_i'Q_iL = M'Q_i'Q_iM, \quad i \in \mathcal{S}_f, \qquad (3.281)$$

consequently, for $i \in \mathcal{S}_f$

$$\text{rank}(L'Q_i'Q_iL) = \text{rank}(M'Q_i') = r_L. \tag{3.282}$$

Thus, $L'Q_i'Q_iL$ is nonsingular for $i \in \mathcal{I}_f$. Therefore, if we define

$$SSH_i = (L'Q_i'\underset{\sim}{\omega}_{fi} - \underset{\sim}{\ell})'(L'Q_i'Q_iL)^{-1}(L'Q_i'\underset{\sim}{\omega}_{fi} - \underset{\sim}{\ell}) \tag{3.283}$$

$(i \in \mathcal{I}_f)$, we obtain

$$SSH_i/\xi_{fi} \sim \chi^2_{r_L}(\Delta_i), \quad i \in \mathcal{I}_f \tag{3.284}$$

where

$$2\xi_{fi}\Delta_i = (L'\underset{\sim}{\tau}_i - \underset{\sim}{\ell})(L'Q_i'Q_iL)^{-1}(L'\underset{\sim}{\tau}_i - \underset{\sim}{\ell}), \quad i \in \mathcal{I}_f. \tag{3.285}$$

Now if $i_0 \in \mathcal{I}_f$ is such that $\xi_{fi_0} = \xi_{fi}$, $i \in \mathcal{I}_f$, then

$$F_{H_i} = \frac{SSH_i/r_L}{MS_{fi_0}} \sim F_{r_L,m_{i_0}}(\Delta_i), \tag{3.286}$$

where, recall, $MS_{fi_0} = \underset{\sim}{\omega}'_{fi_0}\underset{\sim}{\omega}_{fi_0}/m_{i_0}$. Under H_0, $\Delta_i = 0$ and $F_{H_i} \sim F_{r_L,m_{i_0}}$. Thus, F_{H_i} can be used to test $H_0: L'\underset{\sim}{\tau}_i = \underset{\sim}{\ell}$, $i \in \mathcal{I}_f$. We would reject H_0 for large values of the test statistic.

An approximate test. When there does not exist a proper mean square, MS_{fi_0}, with the property that $E[MS_{fi_0}] = \xi_{fi}$ $(i = 0,1,\cdots,\nu\text{-}p;$ $\nu\text{-}p+1 \le i_0 \le \nu)$, then we cannot apply the above methodology to construct a test for $H_0: L'\underset{\sim}{\tau}_i = \underset{\sim}{\ell}$, $i = 0,1,\cdots,\nu\text{-}p$. One common approximate procedure is to develop a "synthetic" mean square that represents a linear combination of the mean squares associated with the random portion of model (3.243) such that its expected value is ξ_{fi} $(i = 0,1,\cdots,\nu\text{-}p)$. This is Satterthwaite's procedure. Unfortunately, the numerator and denominator of the resulting quasi F-ratio will not be independent as required $\big($see(3.250)$\big)$. To overcome this difficulty we first make a transformation (utilizing resampling) that will result in a set of mutually independent, (scaled) chi-squared distributed sums of squares. Then (rightly) apply Satterthwaite's procedure to construct the approximate test. We will illustrate the idea by considering a three-way cross-classification design in which factor A is fixed while factors B and C are random.

To begin with recall from (3.238) that

$$\underset{\sim}{u}'_r = (\underset{\sim}{u}'_{\nu-p+1} : \cdots : \underset{\sim}{u}'_{\nu})'.$$

Let

$$\mathcal{I}^c_f = \{0 \leq i \leq \nu\text{-}p\text{: } \underset{\sim}{u}_i \text{ is part of } \underset{\sim}{v}^c_f\}, \tag{3.287}$$

$\big($see (3.257) and (3.259)$\big)$. Where the vector $\underset{\sim}{v}^c_f$ is defined as follows:

(i) Put $\underset{\sim}{u}_i (i = 0,1,\cdots,\nu\text{-}p)$ in $\underset{\sim}{v}^c_f$ if and only if there does not exist an $i_0 \in \{\nu\text{-}p+1,\cdots,\nu\}$ for which $\delta_i = \delta_{i_0}$ $\big(\delta_k$, $k = 0,1,\cdots,\nu$, is defined in (3.217)$\big)$.

Thus, $\mathcal{I}^c_f$ identifies the subscripts of the $\underset{\sim}{u}_i$ $(i = 0,1,\cdots,\nu\text{-}p)$ that are in $\underset{\sim}{v}^c_f$. Note that every $\underset{\sim}{u}_i$ $(i = 0,1,\cdots,\nu\text{-}p)$ is in either $\underset{\sim}{v}_f$ or $\underset{\sim}{v}^c_f$. Define $\underset{\sim}{x}$ as

$$\underset{\sim}{x} = \begin{bmatrix} \underset{\sim}{v}_f^c \\ \underset{\sim}{u}_r \end{bmatrix}. \tag{3.288}$$

In (3.288), $\underset{\sim}{v}_f^c$ is not independent of $\underset{\sim}{u}_r$ $\left(\text{see } (3.250)\right)$.

Example 3.3

Consider a three-way cross-classification design in which factor A is fixed while factors B and C are random. The model can be expressed as in (3.260). In the notation of Section 3.2.2 we have $\nu = 7$ and $p = 6$. The variance components are σ_2^2, $\sigma_3^2, \cdots, \sigma_7^2$, and σ_ϵ^2. Table 3.4 summarizes the pertinent information needed in the determination of $\underset{\sim}{x}$.

From Table 3.4 we see that

$$\underset{\sim}{v}_f^c = [\underset{\sim}{u}_0' : \underset{\sim}{u}_1'] = \underset{\sim}{u}_f, \tag{3.289}$$

where $\underset{\sim}{u}_f$ was defined in (3.245). Thus, in this example, the vector $\underset{\sim}{x}$ is exactly the vector $\underset{\sim}{u}$ $\left(\text{see } (3.243)\right)$.

Now, let $\mathcal{T}$ be the set

$$\mathcal{T} = \mathcal{S}_f^c \cup \{\nu\text{-}p+1, \cdots, \nu\}. \tag{3.290}$$

Also, define

$$\begin{aligned} \eta_{a1} &= c - \sum_{i \in \mathcal{T}^c} m_i \\ \eta_{a2} &= n-2c + \sum_{i \in \mathcal{T}^c} m_i \end{aligned} \tag{3.291}$$

Table 3.4 Information needed to determine the vector $\underset{\sim}{x}$ in (3.288) for a three-way cross-classification model in which Factor A is fixed while Factors B and C are random

i	ψ_i	b_i	W_i (see (3.217))	δ_i (see (3.217))
0	ϕ	abc	$\{2,3,4,5,6,7\}$	$ac\sigma_2^2+c\sigma_3^2+ab\sigma_4^2+b\sigma_5^2+a\sigma_6^2+\sigma_7^2$
1	$\{i\}$	bc	$\{3,5,7\}$	$c\sigma_3^2+b\sigma_5^2+\sigma_7^2$
2	$\{j\}$	ac	$\{2,3,6,7\}$	$ac\sigma_2^2+c\sigma_3^2+a\sigma_6^2+\sigma_7^2$
3	$\{i,j\}$	c	$\{3,7\}$	$c\sigma_3^2+\sigma_7^2$
4	$\{k\}$	ab	$\{4,5,6,7\}$	$ab\sigma_4^2+b\sigma_5^2+a\sigma_6^2+\sigma_7^2$
5	$\{i,k\}$	b	$\{5,7\}$	$b\sigma_5^2+\sigma_7^2$
6	$\{j,k\}$	a	$\{6,7\}$	$a\sigma_6^2+\sigma_7^2$
7	$\{i,j,k\}$	1	$\{7\}$	σ_7^2

where $\mathcal{T}^c$ is the complement of $\mathcal{T}$ with respect to $\{0,1,\cdots,\nu\}$. Note that $\eta_{a1}+\eta_{a2}=n-c=\text{rank}(R)$. Also, $\eta_{a2}>0$ by (3.195). The matrix R is defined in (3.226). We now introduce a transformation that will result in a set of independent, (scaled) chi-squared distributed sums of squares. The development of this transformation is similar to the development that led up to (3.265) and is thus only outlined.

Define the random vector $\underset{\sim}{\omega}_a$ as

$$\underset{\sim}{\omega}_a = \underset{\sim}{x} + (\lambda_{a,\max} I_{\eta_{a1}} - G_a)^{\frac{1}{2}} C'_{a1} \underset{\sim}{y} \qquad (3.292)$$

where $G_a = Q_x K Q_x'$ and Q_x consists of those Q_i for which $i \in \mathcal{T}$. (In Example 3.3, $Q_x \equiv Q$.) $\lambda_{a,\max}$ is the largest eigenvalue of G_a; and C_{a1} as an $n \times \eta_{a1}$ matrix constructed similarly to C_{f_1} in (3.263), for example.

Partition $\underset{\sim}{x}$ as

$$\underset{\sim}{x} = (\underset{\sim}{x}_i : i \in \mathcal{T}), \tag{3.293}$$

where each $\underset{\sim}{x}_i$ is of dimension m_i, $i \in \mathcal{T}$. We note that $\{\underset{\sim}{x}_i : i \in \mathcal{T}\}$ represents a subset of $\{\underset{\sim}{u}_0, \underset{\sim}{u}_1, \cdots, \underset{\sim}{u}_\nu\}$. Partition $\underset{\sim}{\omega}_a$ just like $\underset{\sim}{x}$.

That is

$$\underset{\sim}{\omega}_a = (\underset{\sim}{\omega}_{ai} : i \in \mathcal{T}), \tag{3.294}$$

where $\underset{\sim}{\omega}_{ai}$ is of order $m_i \times 1$, $i \in \mathcal{T}$. We then have

Theorem 3.9

(a) $$E(\underset{\sim}{\omega}_{ai}) = \begin{cases} Q_i \underset{\sim}{\tau}_i, & i \in \mathcal{S}_f^c \\ \underset{\sim}{0}, & i \in \{\nu-p+1, \cdots, \nu\}, \end{cases} \tag{3.295}$$

(b) $\{\underset{\sim}{\omega}_{ai} : i \in \mathcal{T}\}$ consists of independently distributed normal random vectors with $\underset{\sim}{\omega}_{ai}$ having variance-covariance matrix

$$\mathrm{Var}(\omega_{ai}) = (\delta_i + \lambda_{a,\max}\sigma_\epsilon^2) I_{m_i}, \quad i \in \mathcal{T}.$$

Proof:

The proof of Theorem 3.9 is similar to that of Theorem 3.8 and is thus omitted. □

In order to construct an approximate test of the testable hypothesis $H_0: L'\underset{\sim}{\tau}_i = \underset{\sim}{\ell}$ vs. $H_a: L'\underset{\sim}{\tau}_i \neq \underset{\sim}{\ell}$, $i \in \mathcal{I}_f^c$, we apply Satterthwaite's procedure (in the usual manner) to the set of vectors $\{\underset{\sim}{\omega}_{ai} : i \in \mathcal{T}\}$.

3.3.5 Reduction of the Tests When the Design is Balanced

If the design is balanced, that is, if $n_\tau \equiv N$ for all $\tau \in T$ (see (3.190) and (3.191)), then the c×c matrix K given in (3.211) becomes

$$K = \frac{1}{N} I_c.$$

Consequently, the matrices G_r, G_f, and G_a become, respectively, $\frac{1}{N} I_{\eta_{r1}}$, $\frac{1}{N} I_{\eta_{f1}}$, and $\frac{1}{N} I_{\eta_{a1}}$. (See (3.223), (3.266), and (3.292).) Hence, each matrix has $\frac{1}{N}$ as its largest eigenvalue. Thus, when the design is balanced, $\underset{\sim}{\omega}_r$, $\underset{\sim}{\omega}_f$, and $\underset{\sim}{\omega}_a$ reduce to $\underset{\sim}{u}_r$, $\underset{\sim}{v}$, and $\underset{\sim}{x}$, respectively. (see (3.237), (3.265), and (3.292), respectively.) Recall that $\underset{\sim}{u}_r = Q_r \underset{\sim}{\bar{y}}$ (see (3.220) and (3.221)). Similarly, $\underset{\sim}{v} = Q_v \underset{\sim}{\bar{y}}$ (see (3.271)) and $\underset{\sim}{x} = Q_x \underset{\sim}{\bar{y}}$ (see (3.288) and the discussion after (3.292)).

Now, from (3.243), and (3.252), and the fact that the matrix Q in (3.214) is orthogonal, we have

$$\underset{\sim}{u}'\underset{\sim}{u} = \underset{\sim}{\bar{y}}' Q' Q \underset{\sim}{\bar{y}}$$

$$= \underset{\sim}{\bar{y}}' \left(\sum_{i=0}^{\nu} Q_i' Q_i \right) \underset{\sim}{\bar{y}}$$

$$= \sum_{i=0}^{\nu} \underset{\sim}{\bar{y}}' P_i \underset{\sim}{\bar{y}}. \tag{3.296}$$

From Khuri's (1990) Lemma 3.2 and (3.208) we have

$$\underset{\sim}{\bar{y}}' P_i \underset{\sim}{\bar{y}} = \underset{\sim}{\bar{y}}' \left[\sum_{j=0}^{\nu} (\lambda_{ij}/b_j) A_j \right] \underset{\sim}{\bar{y}}$$

$$= \underset{\sim}{\bar{y}}' \left[\sum_{j=0}^{\nu} (\lambda_{ij}/b_j) \left(\bigotimes_{\ell=1}^{s-1} M_{j\ell} \right) \right] \underset{\sim}{\bar{y}}, \tag{3.297}$$

$(i = 0, 1, \cdots, \nu)$. A_j $(j = 0, 1, \cdots, \nu)$ was defined in (3.207). Here λ_{ij} is a known constant whose possible values are -1, 0, and 1 (see Khuri (1982) for a more detailed definition of the λ_{ij}). Since, the design is balanced

$$\underset{\sim}{\bar{y}} = \tfrac{1}{N} (I_c \otimes \underset{\sim}{1}_N') \underset{\sim}{y}.$$

Therefore (3.297) can also be expressed as

$$\underset{\sim}{\bar{y}}' P_i \underset{\sim}{\bar{y}} = \frac{1}{N^2} \underset{\sim}{y}' (I_c \otimes \underset{\sim}{1}_N) \left[\sum_{j=0}^{\nu} (\lambda_{ij}/b_j) \left(\bigotimes_{\ell=1}^{s-1} M_{j\ell} \right) \right] (I_c \otimes \underset{\sim}{1}_N') \underset{\sim}{y}$$

$$= \frac{1}{N^2} \underset{\sim}{y}' \left[\sum_{j=0}^{\nu} (\lambda_{ij}/b_j) \left\{ \left(\bigotimes_{\ell=1}^{s-1} M_{j\ell} \right) \otimes J_N \right\} \right] \underset{\sim}{y}$$

$$= \frac{1}{N} \underset{\sim}{y}' \left[\sum_{j=0}^{\nu} (\lambda_{ij}/N b_j) \left\{ \left(\bigotimes_{\ell=1}^{s-1} M_{j\ell} \right) \otimes J_N \right\} \right] \underset{\sim}{y} \tag{3.298}$$

$(i=0,1,\cdots,\nu)$. Thus, when the design is balanced, $N\bar{\underset{\sim}{y}}'P_0\bar{\underset{\sim}{y}}$, $N\bar{\underset{\sim}{y}}'P_1\bar{\underset{\sim}{y}},\cdots,N\bar{\underset{\sim}{y}}'P_\nu\bar{\underset{\sim}{y}}$ reduce to the usual balanced ANOVA sums of squares.

Recalling the expressions for $\underset{\sim}{u}_r$, $\underset{\sim}{v}$, and $\underset{\sim}{x}$ given above we see that when the design is balanced

$$\underset{\sim}{\omega}_r'\underset{\sim}{\omega}_r = \underset{\sim}{u}_r'\underset{\sim}{u}_r = \sum_{i=\nu-p+1}^{\nu} \bar{\underset{\sim}{y}}'P_i\bar{\underset{\sim}{y}},$$

$$\underset{\sim}{\omega}_f'\underset{\sim}{\omega}_f = \underset{\sim}{v}'\underset{\sim}{v} = \sum_{i\in\mathcal{S}} \bar{\underset{\sim}{y}}'P_i\bar{\underset{\sim}{y}},$$

$$\underset{\sim}{\omega}_a'\underset{\sim}{\omega}_a = \underset{\sim}{x}'\underset{\sim}{x} = \sum_{i\in\mathcal{T}} \bar{\underset{\sim}{y}}'P_i\bar{\underset{\sim}{y}}.$$

Consequently, since $\{\nu-p+1,\cdots,\nu\}$, $\mathcal{S}$, and $\mathcal{T}$ are all subsets of $\{0,1,\cdots,\nu\}$, the exact tests developed for the random and fixed effects reduce to the usual ANOVA tests associated with a balanced mixed model of the form given in (3.189). The approximate test reduces to the approximate test (based on Satterthwaite's procedure) that would have been obtained if the design had been balanced to begin with.

3.3.6 The Power of the Exact Tests

In this section we examine the power of the exact tests (when such exist). To begin with we consider the power of the exact test associated with the i^{th} variance component $(i=\nu-p+1,\cdots,\nu)$.

The power of the exact test associated with the hypothesis $H_0:\sigma_i^2=0$, $i=\nu-p+1,\cdots,\nu$. For sake of convenience, let $SS_{r,\nu+1}\equiv SSE_{r2}$, $\delta_{\nu+1}\equiv 0$, and $\eta_{r2}\equiv m_{\nu+1}/\lambda_{r,\max}$ (see (3.232) and

(3.235)). We wish to test $H_0 : \sigma_i^2 = 0$ vs. $H_a : \sigma_i^2 \neq 0$, $i = \nu\text{-}p+1, \cdots, \nu$. Suppose that there exists a $j(\neq i) \in \{\nu\text{-}p+1, \cdots, \nu+1\}$ such that under H_0,

$$E\left[\frac{SS_{ri}}{m_i}\right] = \delta_j + \lambda_{r,\max}\sigma_\epsilon^2 = E\left[\frac{SS_{rj}}{m_j}\right]$$

as in (3.242). Then the appropriate test statistic is given by

$$F_{ri} = \frac{SS_{ri}/m_i}{SS_{rj}/m_j} \underset{H_0}{\sim} F_{m_i, m_j}.$$

Under $H_a : \sigma_i^2 \neq 0$, we have

$$\frac{SS_{ri}/m_i}{\sum\limits_{k \in W_i} b_k\sigma_k^2 + \lambda_{r,\max}\sigma_\epsilon^2} \times \frac{\sum\limits_{k \in W_j} b_k\sigma_k^2 + \lambda_{r,\max}\sigma_\epsilon^2}{SS_{rj}/m_j} \sim F_{m_i, m_j}.$$

This follows from (3.217) and Theorem 3.7. Hence if Ψ_{ri} represents the power of the test for σ_i^2, $i = \nu\text{-}p+1, \cdots, \nu$, then

$$\begin{aligned}\Psi_{ri} &= P(F_{ri} \geq F_{\alpha, m_i, m_j} | H_a) \\ &= P\left(F_{m_i, m_j} \geq \frac{1}{1+b_i\theta_{ri}} F_{\alpha, m_i, m_j}\right) \qquad (3.299)\end{aligned}$$

where

$$\theta_{ri} = \frac{\sigma_i^2}{\sum\limits_{k \in W_j} b_k\sigma_k^2 + \lambda_{r,\max}\sigma_\epsilon^2}, \qquad (3.300)$$

($i = \nu\text{-}p+1, \cdots, \nu$; $j(\neq i) \in \{\nu\text{-}p+1, \cdots, \nu+1\}$ such that $\delta_i = \delta_j$). Since Ψ_{ri} increases monotonically with θ_{ri}, $i = \nu\text{-}p+1, \cdots, \nu$, it follows that it is a monotone decreasing function of $\lambda_{r,\max}$ (for fixed ratios, $\sigma_i^2/\sigma_\epsilon^2$ $(i=\nu-p+1,\cdots,\nu)$, of the variance components). It is thus desirable for $\lambda_{r,\max}$ to be small.

The power of the exact test associated with the hypothesis $H_0\colon L'\underline{\tau}_i=\underline{\ell}$, $i\in\mathcal{I}_f$. To test $H_0\colon L'\underline{\tau}_i=\underline{\ell}$, $i\in\mathcal{I}_f$, the appropriate test statistic was (see (3.286))

$$F_{Hi}=\frac{SSH_i/r_L}{MS_{fi_0}},\quad i\in\mathcal{I}_f.$$

Recall that $r_L=\mathrm{rank}(L)$. Also, SSH_i $(i\in\mathcal{I}_f)$ was defined in (3.283). Furthermore, $i_0\in\mathcal{I}_r$ is such that $E[MS_{fi_0}]=\xi_{fi}$ (see (3.259), (3.273), and (3.276)). Now, under $H_a\colon L'\underline{\tau}_i\neq\underline{\ell}$, F_{Hi} has a noncentral F-distribution with noncentrality parameter $\Delta_i(i\in\mathcal{I}_f)$ (see (3.285)). Thus, if Ψ_{fi} is the power associated with this test, then

$$\begin{aligned}\Psi_{fi}&=P(F_{Hi}>F_{\alpha,r_L,m_{i_0}}|H_a)\\&=P(F_{r_L,m_{i_0}}(\Delta_i)>F_{\alpha,r_L,m_{i_0}}),\quad i\in\mathcal{I}_f.\end{aligned}\tag{3.301}$$

Now for $i\in\mathcal{I}_f$, Ψ_{fi} monotonically increases in Δ_i (see Graybill (1976), p. 130). However Δ_i monotonically decreases in ξ_{fi} (recall (3.285)). Since $\xi_{fi}=\delta_i+\lambda_{f,\max}\sigma_\epsilon^2$, Δ_i monotonically decreases in $\lambda_{f,\max}$, $i\in\mathcal{I}_f$. Therefore it is desirable to have $\lambda_{f,\max}$ small.

The design parameters $\lambda_{r,\max}$ and $\lambda_{f,\max}$. The following lemma provides upper and lower bounds for $\lambda_{r,\max}$ and $\lambda_{f,\max}$.

Lemma 3.11

Let $n^{(1)} = \min_{\tau \in T} n_\tau$ and recall the definitions of η_{r1} and η_{f1} in (3.232) and (3.264), respectively. Then,

(a) $\frac{1}{\eta_{r1}}\left[\frac{c-1}{c}\sum_{\tau \in T}\frac{1}{n_\tau} - \frac{c - n_{r1} - 1}{n^{(1)}}\right] \le \lambda_{r,\max} \le \frac{1}{n^{(1)}}.$

(b) $\frac{1}{\eta_{f1}}\left[\sum_{\tau \in T}\frac{1}{n_\tau} - \frac{c - \eta_{f1}}{n^{(1)}}\right] \le \lambda_{f,\max} \le \frac{1}{n^{(1)}}.$

Proof:

(a) Recall that $e_{\max}(A)$ denotes the largest eigenvalue of the matrix A. Then, from (3.223),

$$\lambda_{r,\max} = e_{\max}(G_r) = e_{\max}(Q_r K Q_r')$$

$$\le e_{\max}(K) e_{\max}(Q_r' Q_r)$$

$$= e_{\max}(K) e_{\max}(Q_r Q_r')$$

$$= e_{\max}(K) \qquad (Q_r \text{ has orthonormal rows})$$

$$= \frac{1}{n^{(1)}}.$$

It is also true that

$$e_{\max}(Q_r K Q_r') \ge \frac{1}{\eta_{r1}} \mathrm{tr}(Q_r K Q_r').$$

The term on the right being the average of the eigenvalues of Q_rKQ_r'. However,

$$\operatorname{tr}(Q_rKQ_r') = \operatorname{tr}(KQ_r'Q_r)$$

$$= \operatorname{tr}\Big(K(I_c - Q_f'Q_f)\Big) \qquad \Big(\text{see } (3.244)\Big).$$

From Lemma 3.8(a) and for $Q_f^* = [Q_1' : \cdots : Q_{\nu-p}']'$ we have

$$Q_f'Q_f = [\tfrac{1}{\sqrt{c}}\underset{\sim}{1}_c : Q_f^{*\prime}] \begin{bmatrix} \frac{1}{\sqrt{c}}\underset{\sim}{1}_c' \\ Q_f^* \end{bmatrix} = \bar{J}_c + Q_f^{*\prime}Q_f^*.$$

Hence

$$\operatorname{tr}\Big(K(I_c - Q_f'Q_f)\Big) = \operatorname{tr}\Big(K(I_c - \bar{J}_c) - KQ_f^{*\prime}Q_f^*\Big)$$

$$= \operatorname{tr}\Big(K(I_c - \bar{J}_c)\Big) - \operatorname{tr}(KQ_f^{*\prime}Q_f^*).$$

Now,

$$\operatorname{tr}\Big(K(I_c - \bar{J}_c)\Big) = \operatorname{tr}(K) - \operatorname{tr}(K\bar{J}_c)$$

$$= \operatorname{tr}(K) - \tfrac{1}{c}\operatorname{tr}(K\underset{\sim}{1}_c\underset{\sim}{1}_c')$$

$$= \operatorname{tr}(K) - \tfrac{1}{c}\underset{\sim}{1}_c'K\underset{\sim}{1}_c$$

$$= \sum_{\tau \in T}\frac{1}{n_\tau} - \frac{1}{c}\sum_{\tau \in T}\frac{1}{n_\tau}$$

$$= \frac{c-1}{c}\sum_{\tau \in T}\frac{1}{n_\tau}.$$

Also, $\operatorname{tr}(KQ_f^{*\prime}Q_f^*) = \operatorname{tr}(Q_f^*KQ_f^{*\prime})$. But, since $Q_f^*Q_f^{*\prime} = I_{c-\eta_{r1}-1}$, it follows that $\frac{1}{n^{(1)}}Q_f^*Q_f^{*\prime} - Q_f^*KQ_f^{*\prime}$ is p.s.d. Consequently,

$$\operatorname{tr}(Q_f^*KQ_f^{*\prime}) \le \frac{1}{n^{(1)}}(c - n_{r1} - 1).$$

Thus

$$\operatorname{tr}(Q_rKQ_r') \ge \frac{c-1}{c} \sum_{\tau \in T} \frac{1}{n_\tau} - \frac{c - n_{r1} - 1}{n^{(1)}},$$

and

$$\lambda_{r,\max} \ge \frac{1}{\eta_{r1}}\left[\frac{c-1}{c} \sum_{\tau \in T} \frac{1}{n_\tau} - \frac{c - \eta_{r1} - 1}{n^{(1)}}\right].$$

(b) $\lambda_{f,\max}$ is the largest eigenvalue of G_f defined in (3.266). Thus

$$\begin{aligned} \lambda_{f,\max} &= e_{\max}(G_f) = e_{\max}(Q_vKQ_v') \\ &\le e_{\max}(K)e_{\max}(Q_v'Q_v) \\ &= \frac{1}{n^{(1)}} \qquad (Q_v \text{ has orthonormal rows}). \end{aligned}$$

On the other hand, it is true that $\lambda_{f,\max}$ is greater than or equal to the average of the eigenvalues of Q_vKQ_v'. That is,

$$e_{\max}(Q_vKQ_v') \ge \frac{1}{\eta_{f1}} \operatorname{tr}(Q_vKQ_v').$$

Now,

$$\mathrm{tr}(Q_V K Q_V') = \mathrm{tr}(K Q_V' Q_V)$$

$$= \mathrm{tr}\Big(K(I - Q_V^{c\prime} Q_V^c)\Big)$$

where Q_V^c consists of all those matrices, Q_i, for which $i \in \mathscr{I}^c$. $\Big($We note that if $Q_V \equiv Q$ then $\mathrm{tr}(Q_V K Q_V') = \mathrm{tr}(K) = \sum_{\tau \in T} \frac{1}{n_\tau}$. And thus $e_{\max}(Q_V K Q_V') \geq \frac{1}{\eta_{f1}} \sum_{\tau \in T} \frac{1}{n_\tau}$. Also, in this case $\eta_{f1} \equiv c$ (see (3.264))$\Big)$. Now,

$$\mathrm{tr}\Big(K(I - Q_V^{c\prime} Q_V^c)\Big) = \mathrm{tr}(K) - \mathrm{tr}(Q_V^c K Q_V^{c\prime})$$

$$= \sum_{\tau \in T} \frac{1}{n_\tau} - \mathrm{tr}(Q_V^c K Q_V^{c\prime}).$$

However, $\frac{1}{n^{(1)}} Q_V^c Q_V^{c\prime} - Q_V^c K Q_V^{c\prime}$ is p.s.d. Consequently,

$$\mathrm{tr}(Q_V^c K Q_V^{c\prime}) \leq \frac{1}{n^{(1)}} \mathrm{tr}(Q_V^c Q_V^{c\prime})$$

$$= \frac{1}{n^{(1)}} \; (c - \eta_{f1}).$$

Thus,

$$\mathrm{tr}(Q_V K Q_V') \geq \sum_{\tau \in T} \frac{1}{n_\tau} - \frac{c - \eta_{f1}}{n^{(1)}}.$$

Hence

$$\lambda_{f,\max} = e_{\max}(Q_v K Q_v') \geq \frac{1}{\eta_{f1}}\left[\sum_{\tau \in T} \frac{1}{n_\tau} - \frac{c - \eta_{f1}}{n^{(1)}}\right]. \qquad \square$$

Some remarks are in order:

(i) In an unbalanced random effects model where the imbalance affects the last stage only, such as described in Khuri (1990), $c - \eta_{r1} = 1$. Thus the lower bound given for $\lambda_{r,\max}$ in Lemma 3.11(a) reduces to the lower bound reported in Khuri's (1990) Inequality (4.1).

(ii) If Q_0 is not one of the matrices in the definition of Q_v (see (3.271)), then similar to part (a) of this lemma, the lower bound for $\lambda_{f,\max}$ can be improved upon. The improved lower bound can be shown to equal

$$\frac{1}{\eta_{f1}}\left[\frac{c-1}{c} \sum_{\tau \in T} \frac{1}{n_\tau} - \frac{c - \eta_{f1} - 1}{n^{(1)}}\right].$$

This follows similarly to the proof of Lemma 3.11(a).

(iii) Recall that $\lambda_{r,\max}$ is the largest eigenvalue of $Q_r K Q_r'$ while $\lambda_{f,\max}$ is the largest eigenvalue of $Q_v K Q_v'$. Similar to the discussion concerning $\lambda_{\max}$ presented after Lemma 3.7, $\lambda_{r,\max}$ will be invariant to the choice of Q_r and $\lambda_{f,\max}$ will be invariant to the choice of Q_v.

3.3.7 An Alternative Approach to the Analysis of the Unbalanced Mixed Model with Imbalance Affecting the Last Stage Only

Introduction. In this section we reanalyze model (3.189). Instead of making transformations on only portions of $\underset{\sim}{u}$ $\big($as in (3.237), (3.265), and (3.292)$\big)$, we now transform all of $\underset{\sim}{u}$. We will again utilize resampling in this transformation. The result will be a set of sums of squares which are independent and distributed as scaled chi-squared variates analogous to those in a balanced mixed model.

This approach has advantages and disadvantages. One advantage is that only a single transformation is needed to reduce the analysis to that of a balanced mixed model. Thus exact tests (when they exist) for the variance components and for estimable linear functions of the fixed effects are easily constructed (as they are in the balanced case). Also, when no exact tests exist, Satterthwaite's procedure is readily applied in the usual way.

A disadvantage to this approach is that the power of the resulting exact tests cannot be any greater than the power of the exact tests derived in the previous sections. In fact, we will show that the power of the exact tests resulting from this new approach is equivalent to the power of the tests we would have obtained if we had just simply deleted observations until each cell contained $n^{(1)} = \min_{\tau \in T} n_\tau$ observations.

This disadvantage is the principal reason why we recommend adopting this approach only when a few of the cell frequencies are large relative to $n^{(1)}$. In situations where only a few of the cells contain $n^{(1)}$ observations and $n^{(1)}$ is small relative to the other cell counts, we recommend the use of the procedures developed in the previous sections.

One final remark: Unfortunately some notations will need to be reused. However, this should pose no difficulty since it will be clear from the context when a symbol is uniquely defined for this section or when it has taken on the same meaning it had in some previous section.

Utilizing resampling. Recall the definition of $\underset{\sim}{u}$ given in (3.243), namely $\underset{\sim}{u} = Q\bar{\underset{\sim}{y}}$ $\big($Q is defined in (3.214)$\big)$. From (3.246) we have

$$\underset{\sim}{u} \sim N\left(\begin{bmatrix} Q_f H \underset{\sim}{g} \\ \underset{\sim}{0} \end{bmatrix}, \bigoplus_{i=0}^{\nu} \delta_i I_{m_i} + \sigma_\epsilon^2 G\right),$$

where Q_f is given in (3.244) and G is defined in (3.247). The δ_i $(i = 0, 1, \cdots, \nu)$ are defined in (3.217).

The matrix R $\big($see (3.226)$\big)$ can be written as

$$R = C \Lambda C' \tag{3.302}$$

where C is an orthogonal matrix and Λ is a diagonal matrix whose diagonal entries consist of n-c ones and c zeros. We can partition C and Λ as

$$\begin{aligned} C &= [C_1 : C_2 : C_3] \\ \Lambda &= \operatorname{diag}(I_c, I_{n-2c}, 0) \end{aligned} \tag{3.303}$$

where C_1, C_2, and C_3 are matrices of orders $n \times c$, $n \times (n-2c)$, and $n \times c$, respectively. We must necessarily assume

$$n > 2c. \tag{3.304}$$

This assumption is more stringent than the assumption in (3.195) in the sense that if (3.304) holds then (3.195) will hold. But the opposite may not be true. Even if (3.195) holds, (3.304) may not hold.

Since C is orthogonal we have

$$\begin{aligned} C_i'C_i &= I \qquad (i = 1,2,3) \\ C_j'C_j &= 0 \qquad (i \neq j). \end{aligned} \tag{3.305}$$

Consequently,

$$R = C_1C_1' + C_2C_2'. \tag{3.306}$$

Also,

$$\begin{aligned} SSE = \underset{\sim}{y}'R\underset{\sim}{y} &= \underset{\sim}{y}'C_1C_1'\underset{\sim}{y} + \underset{\sim}{y}C_2C_2'\underset{\sim}{y} \\ &= SSE_1 + SSE_2, \end{aligned} \tag{3.307}$$

where

$$SSE_i = \underset{\sim}{y}'C_iC_i'\underset{\sim}{y}, \quad i = 1,2. \tag{3.308}$$

Furthermore $\underset{\sim}{\bar{y}}$, SSE_1, and SSE_2 are mutually independent and

$$SSE_1/\sigma_\epsilon^2 \sim \chi_c^2$$
$$SSE_2/\sigma_\epsilon^2 \sim \chi_{n-2c}^2. \qquad (3.309)$$

Now, we utilize resampling to transform the vector $\underset{\sim}{u}$ into a random vector that has a diagonal variance-covariance matrix. We do this by considering the $c \times 1$ random vector $\underset{\sim}{\omega}$ defined as

$$\underset{\sim}{\omega} = \underset{\sim}{u} + (\lambda_{\max} I_c - G)^{\frac{1}{2}} C_1' \underset{\sim}{y}, \qquad (3.310)$$

where $\lambda_{\max}$ is the largest eigenvalue of G. Because $\lambda_{\max} I_c - G$ is p.s.d., $(\lambda_{\max} I_c - G)^{\frac{1}{2}}$ is well-defined with eigenvalues equal to the square roots of the eigenvalues of $\lambda_{\max} I_c - G$. We note that in this case

$$\begin{aligned}\lambda_{\max} &= e_{\max}(G) = e_{\max}(QKQ') \\ &= e_{\max}(K) \qquad (Q \text{ is orthogonal}) \\ &= \frac{1}{n^{(1)}} \qquad (n^{(1)} = \min_{\tau \in T} n_\tau).\end{aligned} \qquad (3.311)$$

Partition $\underset{\sim}{\omega}$ just like $\underset{\sim}{u}$ in (3.245). That is

$$\underset{\sim}{\omega} = (\underset{\sim}{\omega}_f' : \underset{\sim}{\omega}_r')' = (\underset{\sim}{\omega}_0' : \cdots : \underset{\sim}{\omega}_{\nu-p}' : \underset{\sim}{\omega}_{\nu-p+1}' : \cdots : \underset{\sim}{\omega}_\nu')' \qquad (3.312)$$

where each $\underset{\sim}{\omega}_i$ is of order $m_i \times 1$, $i=0,1,\cdots,\nu$. The next theorem provides some distribution properties for the $\underset{\sim}{\omega}_i$, $i=0,1,\cdots,\nu$.

Theorem 3.10

(a) $$E(\underset{\sim}{\omega}_i) = \begin{cases} Q_i H\underset{\sim}{g}, & i=0,1,\cdots,\nu\text{-}p \\ \underset{\sim}{0}, & i=\nu\text{-}p+1,\cdots,\nu. \end{cases}$$

(b) $\underset{\sim}{\omega}_0,\underset{\sim}{\omega}_1,\cdots,\underset{\sim}{\omega}_\nu$ are normally distributed random vectors with $\underset{\sim}{\omega}_i$ having variance-covariance matrix

$$\mathrm{Var}(\underset{\sim}{\omega}_i) = (\delta_i + \frac{1}{n^{(1)}}\sigma_\epsilon^2) I_{m_i}, \qquad i=0,1,\cdots,\nu.$$

(c) The vector $\underset{\sim}{\omega}$ is independent of SSE_2.

Proof:

The proof of parts (a) and (b) follows closely to that given in Theorem 3.8. The proof of part (c) closely mimics that given for Theorem 3.7(c). □

Recall from (3.256) that $Q_i H\underset{\sim}{g} = Q_i \underset{\sim}{\tau}_i$, $i=0,1,\cdots,\nu\text{-}p$, where the vector $\underset{\sim}{\tau}_i$ is defined in Lemma 3.10(c).

The vector $\underset{\sim}{\omega}$. Recalling Lemma 3.9 and that

$$\underset{\sim}{y} = X\underset{\sim}{g} + Z\underset{\sim}{h} + \underset{\sim}{\epsilon} \qquad \Big(\text{see } (3.196)\Big),$$

$$\bar{\underset{\sim}{y}} = H\underset{\sim}{g} + F\underset{\sim}{h} + \bar{\underset{\sim}{\epsilon}} \qquad \Big(\text{see } (3.206)\Big),$$

and

$$\underset{\sim}{u} = Q\bar{\underset{\sim}{y}} \qquad \left(\text{see } (3.243)\right),$$

we see that the vector $\underset{\sim}{\omega}$ $\left(\text{in } (3.310)\right)$ can be expressed as

$$\underset{\sim}{\omega} = Q\bar{\underset{\sim}{y}} + \left(\frac{1}{n^{(1)}}I_c - G\right)^{\frac{1}{2}}C_1'(X\underset{\sim}{g} + Z\underset{\sim}{h} + \underset{\sim}{\epsilon})$$

$$= QH\underset{\sim}{g} + QF\underset{\sim}{h} + Q\bar{\underset{\sim}{\epsilon}} + \left(\frac{1}{n^{(1)}}I_c - G\right)^{\frac{1}{2}}C_1'\underset{\sim}{\epsilon}.$$

The last equality follows since $\text{range}(C_1) \subset \text{range}(R)$ and consequently $C_1'X = C_1'Z = 0$. Now, let

$$H_\omega = QH = \begin{pmatrix} Q_f H \\ Q_r H \end{pmatrix} = \begin{pmatrix} Q_f H \\ 0 \end{pmatrix} \qquad (3.313)$$

$$F_\omega = QF = \begin{pmatrix} Q_f F \\ Q_r F \end{pmatrix} \qquad (3.314)$$

$$\underset{\sim}{\epsilon}_\omega = Q\bar{\underset{\sim}{\epsilon}} + \left(\frac{1}{n^{(1)}}I_c - G\right)^{\frac{1}{2}}C_1'\underset{\sim}{\epsilon}. \qquad (3.315)$$

In (3.313) recall from Lemma 3.8(c) that $Q_i H = 0$ for $i = \nu\text{-}p\text{+}1, \cdots, \nu$. Also Q_f was defined in (3.244) while Q_r was given in (3.220). Therefore we can express $\underset{\sim}{\omega}$ as

$$\underset{\sim}{\omega} = H_\omega \underset{\sim}{g} + F_\omega \underset{\sim}{h} + \underset{\sim}{\epsilon}_\omega. \qquad (3.316)$$

Clearly $E(\underset{\sim}{\epsilon}_\omega) = \underset{\sim}{0}$. Furthermore, since $\bar{\underset{\sim}{\epsilon}} = D\underset{\sim}{\epsilon}$ $\left(\text{see } (3.200)\right)$, it must be that $\bar{\underset{\sim}{\epsilon}}$ and $C_1'\underset{\sim}{\epsilon}$ are independent. This follows since $DR = 0$ $\left(\text{Lemma 3.9(b)}\right)$ and $\text{range}(C_1) \subset \text{range}(R)$. Consequently,

$$\mathrm{Var}(\underset{\sim}{\epsilon}_{\omega}) = \mathrm{Q}\,\mathrm{Var}(\overline{\underset{\sim}{\epsilon}})\mathrm{Q}' + (\frac{1}{n^{(1)}}\mathrm{I}_c - \mathrm{G})^{\frac{1}{2}}\mathrm{C}_1'\mathrm{Var}(\underset{\sim}{\epsilon})\mathrm{C}_1(\frac{1}{n^{(1)}}\mathrm{I}_c - \mathrm{G})^{\frac{1}{2}}$$

$$= \sigma_\epsilon^2\mathrm{QKQ}' + \sigma_\epsilon^2(\frac{1}{n^{(1)}}\mathrm{I}_c - \mathrm{G})$$

$$= \frac{1}{n^{(1)}}\sigma_\epsilon^2\mathrm{I}_c.$$

The last equality follows since $\mathrm{QKQ}' = \mathrm{G}$. Thus model (3.316) describes a balanced mixed-model. Letting $\xi_i = \delta_i + \frac{1}{n^{(1)}}\sigma_\epsilon^2$, $i = 0,1,\cdots,\nu$, we obtain from Theorem 3.10 that

$$\underset{\sim}{\omega} \sim \mathrm{N}\Big(\mathrm{H}_\omega\underset{\sim}{g},\ \bigoplus_{i=0}^{\nu}\xi_i\mathrm{I}_{m_i}\Big). \tag{3.317}$$

Let

$$\mathrm{SS}_i = \underset{\sim}{\omega}_i'\underset{\sim}{\omega}_i, \quad i = 0,1,\cdots,\nu, \tag{3.318}$$

then, from Theorem 3.10, we have:

$$\text{(i)}\quad \mathrm{SS}_i/\xi_i \sim \chi^2_{m_i}(\Xi_i), \quad i = 0,1,\cdots,\nu \tag{3.319}$$

where

$$\Xi_i = \begin{cases} \frac{1}{2\xi_i}\underset{\sim}{\tau}_i'\mathrm{Q}_i'\mathrm{Q}_i\underset{\sim}{\tau}_i, & i = 0,1,\cdots,\nu\text{-p} \\ 0, & i = \nu\text{-p}+1,\cdots,\nu. \end{cases} \tag{3.320}$$

(ii) For $MS_i = SS_i/m_i, \quad i = 0,1,\cdots,\nu$

$$E(MS_i) = \begin{cases} \xi_i + \frac{1}{m_i}\tau_i' Q_i' Q_i \tau_i, & i = 0,1,\cdots,\nu\text{-}p \\ \xi_i, & i = \nu\text{-}p+1,\cdots,\nu. \end{cases} \tag{3.321}$$

Thus, $SS_0, SS_1, \cdots, SS_\nu$, and SSE_2 act like sums of squares in an ANOVA table for a balanced mixed model (see Khuri (1982), pp. 2915-2917). That is, analyses concerning the variance components and estimable linear functions of the fixed effects can proceed using these sums of squares just like in a balanced data situation. In fact, if the design was balanced, then similar to Section 3.3.5 we can show that $NSS_0, NSS_1, \cdots, NSS_\nu$ reduce to the usual balanced ANOVA sums of squares. (Recall that N is the common last stage cell frequency.)

The variance components. Using $SS_{\nu-p+1}, \cdots, SS_\nu$, and SSE_2 we can proceed to test hypotheses concerning the variance components similarly to the methodology presented in Section 3.3.3. Suppose we wish to test $H_0: \sigma_i^2 = 0$ vs. $H_a: \sigma_i^2 \neq 0$, $i = \nu\text{-}p+1, \cdots, \nu$, then the power Ψ_i, of the exact test (assuming such exists based on $SS_{\nu-p+1}, \cdots, SS_\nu$, and SSE_2) can be expressed as (take $SS_{\nu+1} \equiv SSE_2$, $\delta_{\nu+1} \equiv 0$, $m_{\nu+1} \equiv \frac{n-2c}{n^{(1)}}$)

$$\Psi_i = P\left(\frac{SS_i/m_i}{SS_j/m_j} \geq F_{\alpha, m_i, m_{i_0}} | H_a\right) \quad i = \nu\text{-}p+1, \cdots, \nu, \tag{3.322}$$

where SS_j/m_j is such that $E\left[\frac{SS_i}{m_i}\right] = E\left[\frac{SS_j}{m_j}\right]$ when $H_0: \sigma_i^2 = 0$ is true. Under H_a,

$$\frac{SS_i/m_i}{\sum\limits_{k \in W_i} b_k \sigma_k^2 + \frac{1}{n^{(1)}}\sigma_\epsilon^2} \times \frac{\sum\limits_{k \in W_j} b_k \sigma_k^2 + \frac{1}{n^{(1)}}\sigma_\epsilon^2}{SS_j/m_j} \sim F_{m_i, m_j}.$$

Thus, for $i=\nu-p+1,\cdots,\nu$,

$$\Psi_i = P\left(F_{m_i,m_j} \geq \frac{1}{1+b_i\theta_i} F_{\alpha,m_i,m_j}\right) \tag{3.323}$$

where

$$\theta_i = \frac{\sigma_i^2}{\sum_{k\in W_j} b_k\sigma_k^2 + \frac{1}{n^{(1)}}\sigma_\epsilon^2}. \tag{3.324}$$

(Compare (3.223) and (3.224) to their counterparts expressed in (3.299) and (3.300), respectively.)

Now, let us suppose that the design is indeed balanced. That is, suppose $n_\tau \equiv N$ for all $\tau \in T$ (recall (3.191)). To test the above null hypothesis we would form

$$F_{Bi} = \frac{NSS_i/m_i}{NSS_j/m_j} = \frac{SS_i/m_i}{SS_j/m_j}, \tag{3.325}$$

$(i=\nu-p+1,\cdots,\nu)$. (Recall that SS_j/m_j is such that $E\left[\frac{SS_i}{m_i}\right] = E\left[\frac{SS_j}{m_j}\right]$ when $H_0{:}\sigma_i^2 = 0$ holds.) It is readily shown that the power associated with this test, Ψ_{Bi}, is

$$\Psi_{Bi} = P\left(F_{m_i,m_j} \geq \frac{1}{1+b_i\theta_{Bi}} F_{\alpha,m_i,m_j}\right) \tag{3.326}$$

where

$$\theta_{Bi} = \frac{\sigma_i^2}{\sum_{k\in W_j} b_k\sigma_k^2 + \frac{1}{N}\sigma_\epsilon^2} \tag{3.327}$$

$(i = \nu\text{-}p+1, \cdots, \nu)$. Thus, the power of the above exact test for $H_0: \sigma_i^2 = 0$ in the unbalanced case $\bigl($see (3.323)$\bigr)$ is equivalent to the power of the test based on the following procedure:

(i) Randomly delete observations, if necessary, until all τ-cells contain $n^{(1)}$ observations.

(ii) Analyze the resulting data set as if it arose from a balanced mixed-model with $n^{(1)}$ observations per cell.

We would therefore expect the methodology developed above for testing hypotheses concerning the variance components (for unbalanced models) to be inefficient when most of the cell counts are much larger than $n^{(1)}$. Under such situations, $\lambda_{r,\max}$ will typically be less than $1/n^{(1)}$ and thus we would recommend using the procedures developed in Section 3.3.3 $\bigl($see Lemma 3.11(a)$\bigr)$.

Table 3.5 compares the powers of the exact tests for σ_β^2 for four different designs arising from an unbalanced mixed two-way cross-classification model such as described in Gallo and Khuri (1990). (We assume the "α" effect to be fixed and the "β" and "$\alpha\beta$" effects to be random -- their variances being σ_β^2 and $\sigma_{\alpha\beta}^2$.) The power expressions used in the comparisons were derived in (3.299) and (3.323). The values for σ_β^2 and $\sigma_{\alpha\beta}^2$ are the same ones used in Khuri and Littell (1987).

Design I was chosen so that one cell would have a small cell frequency relative to the rest; while Design II was chosen so that $n^{(1)} = 1$. In both cases $\lambda_{r,\max} < 1/n^{(1)}$ so that the testing procedures developed in Section 3.3.3 are to be recommended. Design IV also had

$n^{(1)} = 1$. However, unlike Design II, almost 25% of the cells had only one observation in them. This is why $\lambda_{r,\max} = 1$. In this case we would recommend using the methodology developed in this subsection. Finally, even though $\lambda_{r,\max} < 1/n^{(1)}$ in Design III, the gain in power is not worth the extra effort necessary to develop the test statistics in Section 3.3.3. Thus in "nearly balanced" designs we would again recommend using the procedures developed in this section.

We should point out that the unbalanced mixed two-way cross-classification model is a very special case of the general unbalanced mixed-model (with imbalance affecting the last stage only). Thus, we may need to modify our recommendations when more general models are considered. We make two final remarks:

(i) In Section 3.3.4 we described a procedure for constructing exact tests (when such exist) for testing a testable hypotheses concerning estimable linear functions of the fixed effects. The power of these tests was derived in (3.301). Recall that the power is a monotonically decreasing function of ξ_{fi} (see Section 3.3.6). Exact tests for the fixed effects can also be constructed by utilizing $SS_0, \cdots, SS_\nu$ $\big($see (3.318)$\big)$. As in Section 3.3.6, the power of these tests will be a monotonically decreasing function of $\xi_i \equiv \delta_i + 1/n^{(1)}\sigma_\epsilon^2$. Since $\xi_{fi} \equiv \delta_i + \lambda_{f,\max}\sigma_\epsilon^2$ and $\lambda_{f,\max} \leq 1/n^{(1)}$ by Lemma 3.11(b), we can conclude, similar to the recommendations made above, that when most of the cell counts are large relative to $n^{(1)}$ the procedures developed

Table 3.5 A comparison of the powers of the exact tests for σ^2_β in an unbalanced mixed two-way cross-classification model

	Cell counts: row=α effect col=β effect	σ^2_β	$\sigma^2_{\alpha\beta}$	Ψ_{r2} (see (3.299))	Ψ_2 (see (3.323))
		.2	.2	.138	.121
	2 10 9	.2	5.0	.059	.059
I	10 10 9	1.0	1.0	.229	.215
	9 9 10	5.0	.2	.787	.750
		5.0	5.0	.273	.268
				$\lambda_{r,max} = .370$; $1/n^{(1)} = .5$	
		.2	.2	.102	.091
	1 4 4	.2	5.0	.058	.058
II	3 3 4	1.0	1.0	.193	.175
	4 4 3	5.0	.2	.687	.633
		5.0	5.0	.260	.252
				$\lambda_{r,max} = .750$; $1/n^{(1)} = 1.0$	
	5 5 5 5 6	.2	.2	.473	.470
	6 5 6 5 5	.2	5.0	.082	.082
III	6 4 5 6 5	1.0	1.0	.668	.667
	6 6 5 5 6	5.0	.2	.994	.994
	6 5 5 5 6	5.0	5.0	.721	.721
				$\lambda_{r,max} = .2457$; $1/n^{(1)} = .25$	

Table 3.5 – continued

	Cell counts: row=α effect col=β effect	σ^2_β	$\sigma^2_{\alpha\beta}$	Ψ_{r2} (see (3.299))	Ψ_2 (see (3.323))
	9 2 9 1 2	.2	.2		
	10 1 2 9 10	.2	5.0	In this case the powers are identical since $\lambda_{r,\max} = 1/n^{(1)} = 1$.	
IV	1 8 1 2 2	1.0	1.0		
	9 10 1 9 3	5.0	.2		
	8 3 2 10 1	5.0	5.0		

in Section 3.3.4 should be used. Otherwise construct exact tests by utilizing the sums of squares developed in this section.

(ii) Our methodology has an advantage over the methodology based on deleting observations until all τ-cells contain $n^{(1)}$ observations (as described earlier) in that it uses all the data in the calculation of the test statistics. This is clearly a desirable feature especially if $n^{(1)} = 1$. For example, if $n^{(1)} = 1$ then the "deleting observations methodology" could not produce a test for $\sigma^2_{\alpha\beta}$ in a two-way cross-classification model, whereas our procedure would allow us to construct such a test.

We now turn our attention to the development of exact simultaneous confidence intervals on estimable linear functions of the fixed effects. Since model (3.316) describes a balanced mixed-model, we will use the results of Khuri (1984) to obtain our intervals.

The fixed effects parameters: Introduction. In Section 3.3.4 we developed exact procedures concerning estimable linear functions of the fixed-effects parameters. The development involved first constructing the vector $\underset{\sim}{v}$ $\left(\text{see (3.257)}\right)$, which consisted of those $\underset{\sim}{u}_i$, $i=0,1,\cdots,\nu\text{-}p$ (and the corresponding $\underset{\sim}{u}_{i_0}$, $i_0=\nu\text{-}p+1,\cdots,\nu$) for which $\delta_i=\delta_{i_0}$, then making a transformation to arrive at $\underset{\sim}{\omega}_f$ $\left(\text{see (3.265)}\right)$. The δ_k, $k=0,1,\cdots,\nu$ were defined in (3.217). The $\underset{\sim}{\omega}_{fi}$, $i\in\mathcal{I}$ $\left(\text{see (3.259) and (3.268)}\right)$ were then utilized in the construction of the exact procedures. The motivation behind the construction of $\underset{\sim}{v}$ came from the procedures used in dealing with estimable linear functions of the fixed-effects parameters in a balanced data situation. For balanced data, exact methods can be derived if a proper mean square can be found in the corresponding ANOVA table that can serve as an "error" term in the F-ratio used for testing a hypothesis concerning the parameters associated with the fixed-effects.

Since $SS_0,\cdots,SS_\nu$ and SSE_2 $\left(\text{see (3.318)}\right)$ mimic the sums of squares associated with a balanced mixed model, we can also develop exact procedures concerning estimable linear functions of the fixed-effects parameters (based on the methodology developed in this section) if we can find a proper mean square that can serve as an "error" term in the F-ratio similarly to the balanced data situation. If no mean square exists that can serve as an "error" term in the F-ratio, then we can apply Satterthwaite's procedure in the usual way to develop approximate methods. However, as pointed out in a recent article by Ames and Webster (1991), in many instances the Satterthwaite estimator of the denominator ("error") degrees of freedom will be positively biased. This will cause the resulting inference to overstate the actual Type I error rate. This same criticism also applies to the approximate method developed in Section 3.3.4. (In certain layouts Ames and Webster (1991) develop an improved estimator of the true denominator degrees of freedom. Details can be obtained from this article.)

For balanced mixed models, Khuri (1984) developed a method for the construction of exact simultaneous confidence intervals on estimable linear functions of the model's fixed-effects parameters in situations when no proper mean square existed which could serve as an "error" term. We now extend his results to the unbalanced mixed model where the imbalance affects the last stage only. The idea is to find a sequence of independent and identically distributed random vectors each having a normal distribution. Furthermore their common mean vector will consist of a vector of estimable linear functions of the fixed-effects parameters. Exact simultaneous confidence intervals on these estimable linear functions can then be obtained by utilizing this sequence of estimators.

The fixed effects parameters: Notation and preliminaries. Recall that $\tau = (k_1, k_2, \cdots, k_{s-1})$ (see (3.191)). By rearranging, if necessary, the order of the elements in τ, we can consider the first s_1 subscripts $(1 \leq s_1 < s-1)$ to be associated with the fixed-effects in (3.316); the next s_2 subscripts $(1 \leq s_2 < s-1)$ are nonnested random subscripts, that is, subscripts associated with those random-effects in (3.316) which do not nest each other and are not nested in any other random effects; and the last s_3 subscripts are associated with random effects nested in at least one of the preceding s_2 effects. We note that

$$s_1 + s_2 + s_3 = s-1. \tag{3.328}$$

Define

$$d_1 = \prod_{i=s_1+1}^{s-1} a_i \tag{3.329}$$

$$d_2 = \begin{cases} \prod_{i=s_1+s_2+1}^{s-1} a_i, & \text{if } s_1+s_2+1 > s-1 \\ 1, & \text{else} \end{cases} \tag{3.330}$$

$$d_3 = \prod_{i=s_1+1}^{s_1+s_2} a_i = d_1/d_2. \tag{3.331}$$

Recall that $c = \prod_{i=1}^{s-1} a_i$ $\left(\text{see } (3.192)\right)$. Also let Λ be the set of vectors of dimension s_2 of the form

$$\Lambda = \{\underset{\sim}{\nu} : \underset{\sim}{\nu} = (i_{s_1+1}, i_{s_2+2}, \cdots, i_{s_1+s_2})'\} \tag{3.332}$$

with

$$i_\ell = 1,2,\cdots,a_\ell, \qquad \ell = s_1+1,\cdots,s_1+s_2.$$

We note that the number of vectors in Λ is d_3. Now, from (3.203) and (3.204) recall that the $c \times c_i$ matrix H_i was defined as

$$H_i = \bigotimes_{\ell=1}^{s-1} L_{i\ell}, \quad i = 0,1,\cdots,\nu$$

where c_i was given in (3.194) and

$$L_{i\ell} = \begin{cases} I_{a_\ell} & \text{for } k_\ell \in \psi_i \\ \underset{\sim}{1}_{a_\ell} & \text{for } k_\ell \in \psi_i^c \end{cases}$$

$(i = 0,1,\cdots,\nu;\quad \ell = 1,2,\cdots,s\text{-}1)$. Hence, for $i = 0,1,\cdots,\nu$,

$$H_i = \Big(\bigotimes_{\ell=1}^{s_1} L_{i\ell}\Big) \otimes \Big(\bigotimes_{\ell=s_1+1}^{s-1} L_{i\ell}\Big). \tag{3.333}$$

Thus, for $i = 0,1,\cdots,\nu\text{-}p$ (that is, H_i is a matrix associated with a fixed-effect) we have

$$\begin{aligned} H_i &= \Big(\bigotimes_{\ell=1}^{s_1} L_{i\ell}\Big) \otimes \Big(\bigotimes_{\ell=s_1+1}^{s-1} L_{i\ell}\Big). \\ &= \Big(\bigotimes_{\ell=1}^{s_1} L_{i\ell}\Big) \otimes \underset{\sim}{1}_{d_1}. \end{aligned} \tag{3.334}$$

This follows since the subscripts associated with the random effects in(3.316) cannot belong to ψ_i, the set of subscripts associated with the fixed i^{th} effect in the model, $i = 0,1,\cdots,\nu\text{-}p$. Note that (3.334) can also be expressed as (see (3.331))

$$H_i = \Big(\bigotimes_{\ell=1}^{s_1} L_{i\ell}\Big) \otimes \underset{\sim}{1}_{d_3} \otimes \underset{\sim}{1}_{d_2},\quad i = 0,1,\cdots,\nu\text{-}p. \tag{3.335}$$

Furthermore, from (3.313), let the $c \times c_i$ matrix $H_{\omega i}$ be

$$H_{\omega i} = QH_i,\quad i = 0,1,\cdots,\nu\text{-}p, \tag{3.336}$$

then

$$H_{\omega i} = Q\left[\left(\overset{s_1}{\underset{\ell=1}{\otimes}} L_{i\ell}\right) \otimes \underset{\sim}{1}_{d_3} \otimes \underset{\sim}{1}_{d_2}\right], \quad i = 0,1,\cdots,\nu\text{-}p. \tag{3.337}$$

For $i = 0,1,\cdots,\nu\text{-}p$ and $\underset{\sim}{\nu} \in \Lambda$ define the $c \times c_i$ matrix $H_i^{\underset{\sim}{\nu}}$ as

$$H_i^{\underset{\sim}{\nu}} = \left(\overset{s_1}{\underset{\ell=1}{\otimes}} L_{i\ell}\right) \otimes \left(\overset{s_1+s_2}{\underset{\ell=s_1+1}{\otimes}} \underset{\sim}{1}_{a_\ell}^{i_\ell}\right) \otimes \underset{\sim}{1}_{d_2}, \tag{3.338}$$

where $\underset{\sim}{1}_{a_\ell}^{i_\ell}$ is a vector of dimension a_ℓ ($\ell = s_1+1,\cdots,s_1+s_2$) whose i_ℓ^{th} entry is unity and the remaining entries zero ($i_\ell = 1,2,\cdots,a_\ell$). Similar to (3.337), define $c \times c_i$ the matrix $H_{\omega_i}^{\underset{\sim}{\nu}}$ as

$$H_{\omega i}^{\underset{\sim}{\nu}} = QH_i^{\underset{\sim}{\nu}}, \quad i = 0,1,\cdots,\nu\text{-}p;\ \underset{\sim}{\nu} \in \Lambda. \tag{3.339}$$

Finally, recall that $A_i = H_i H_i'$, $i = 0,\cdots,\nu$ (see (3.207)). Thus, using (3.208), we have for $i = 0,1,\cdots,\nu\text{-}p$,

$$\begin{aligned} A_i &= \left(\overset{s_1}{\underset{\ell=1}{\otimes}} M_{i\ell}\right) \otimes \left(\overset{s-1}{\underset{\ell=s_1+1}{\otimes}} J_{a_\ell}\right) \\ &= \left(\overset{s_1}{\underset{\ell=1}{\otimes}} M_{i\ell}\right) \otimes J_{d_3} \otimes J_{d_2}. \end{aligned} \tag{3.340}$$

Furthermore, define the $c \times c$ matrix $A_{\omega i}$ as

$$A_{\omega i} = H_{\omega i} H_{\omega i}' = QH_i H_i' Q' = QA_i Q' \tag{3.341}$$

($i = 0,1,\cdots,\nu\text{-}p$). Also, for $i = 0,1,\cdots,\nu\text{-}p$; $\underset{\sim}{\nu} \in \Lambda$ let $A_i^{\underset{\sim}{\nu}} = H_i^{\underset{\sim}{\nu}} H_i^{\underset{\sim}{\nu}\prime}$ and $A_{\omega i}^{\underset{\sim}{\nu}} = H_{\omega i}^{\underset{\sim}{\nu}} H_{\omega i}^{\underset{\sim}{\nu}\prime}$. Then

$$A_i^{\underset{\sim}{\nu}} = \left(\overset{s_1}{\underset{\ell=1}{\otimes}} M_{i\ell}\right) \otimes \left(\overset{s_1+s_2}{\underset{\ell=s_1+1}{\otimes}} J_{a_\ell}^{i_\ell}\right) \otimes J_{d_2} \tag{3.342}$$

and

$$A^{\underset{\sim}{\nu}}_{\omega i} = QH^{\underset{\sim}{\nu}}_{i}H^{\underset{\sim}{\nu}\prime}_{i}Q' = QA^{\underset{\sim}{\nu}}_{i}Q' \tag{3.343}$$

where $J^{i_\ell}_{a_\ell}$ is an $a_\ell \times a_\ell$ matrix whose (i_ℓ, i_ℓ) entry is unity and the remaining entries are zero, $i_\ell = 1,2,\cdots,a_\ell$; $\ell = s_1+1,\cdots,s_1+s_2$. Both $A^{\underset{\sim}{\nu}}_{i}$ and $A^{\underset{\sim}{\nu}}_{\omega i}$ are of order $c \times c$, $i = 0,1,\cdots,\nu\text{-}p$; $\underset{\sim}{\nu} \in \Lambda$.

The fixed-effects parameters: Initial development. From Khuri's (1990) Lemma 3.2 we have for $i = 0,1,\cdots,\nu\text{-}p$

$$P_i = \sum_{j=0}^{\nu}\left(\frac{\lambda_{ij}}{b_j}\right)A_j = \sum_{j=0}^{\nu-p}\left(\frac{\lambda_{ij}}{b_j}\right)A_j. \tag{3.344}$$

The last equality follows since $\lambda_{ij} \equiv 0$ for $i = 0,1,\cdots,\nu\text{-}p$ and $j = \nu\text{-}p+1,\cdots,\nu$ (see Khuri (1982)). The $c \times c$ matrix P_i is associated with the sum of squares for the i^{th} effect in (3.213), $i = 0,1,\cdots,\nu\text{-}p$. Define the $c \times c$ matrix $P_{\omega i}$ as

$$P_{\omega i} = \sum_{j=0}^{\nu-p} \frac{\lambda_{ij}}{b_j}A_{\omega j} = Q\left(\sum_{j=0}^{\nu-p}\frac{\lambda_{ij}}{b_j}A_j\right)Q', \tag{3.345}$$

$(i = 0,1,\cdots,\nu\text{-}p)$. From (3.344), we can equivalently express (3.345) as

$$P_{\omega i} = QP_iQ', \quad i = 0,1,\cdots,\nu\text{-}p. \tag{3.346}$$

Furthermore, define the $c \times c$ matrices $P^{\underset{\sim}{\nu}}_{i}$ and $P^{\underset{\sim}{\nu}}_{\omega i}$ as

$$P^{\underset{\sim}{\nu}}_{i} = \sum_{j=0}^{\nu-p}\left(\frac{\lambda_{ij}}{b_j}\right)A^{\underset{\sim}{\nu}}_{j}, \quad i = 0,1,\cdots,\nu\text{-}p; \; \underset{\sim}{\nu} \in \Lambda \tag{3.347}$$

$$P^{\underset{\sim}{\nu}}_{\omega i} = \sum_{j=0}^{\nu-p}\left(\frac{\lambda_{ij}}{b_j}\right)A^{\underset{\sim}{\nu}}_{\omega j} = QP^{\underset{\sim}{\nu}}_{i}Q', \quad i = 0,1,\cdots,\nu\text{-}p; \ \underset{\sim}{\nu} \in \Lambda. \tag{3.348}$$

The last equality in (3.348) follows from (3.343).

Now, consider the Product $P_{\omega i}H_{\omega i'}$ for $i, i' = 0,1,\cdots,\nu\text{-}p$. From (3.337), (3.340), and (3.345) we obtain

$$P_{\omega i}H_{\omega i'} = QP_iQ'QH_{i'} = QP_iH_{i'}$$

$$= Q\left\{\left(\sum_{j=0}^{\nu-p}\left(\frac{\lambda_{ij}}{b_j}\right)A_j\right)\left(\left(\bigotimes_{\ell=1}^{s_1}L_{i'\ell}\right)\otimes \underset{\sim}{1}_{d_1}\right)\right\}$$

$$= Q\left\{\left[\left(\sum_{j=0}^{\nu-p}\frac{\lambda_{ij}}{b_j}\bigotimes_{\ell=1}^{s_1}M_{j\ell}\right)\otimes J_{d_1}\right]\left[\left(\bigotimes_{\ell=1}^{s_1}L_{i'\ell}\right)\otimes \underset{\sim}{1}_{d_1}\right]\right\}$$

$$= d_1Q\left\{\left(\sum_{j=0}^{\nu-p}\frac{\lambda_{ij}}{b_j}\bigotimes_{\ell=1}^{s_1}M_{j\ell}L_{i'\ell}\right)\otimes \underset{\sim}{1}_{d_1}\right\}. \tag{3.349}$$

Similarly, from (3.337), (3.342), and (3.347) we have

$$P^{\underset{\sim}{\nu}}_{\omega i}H_{\omega i'} = QP^{\underset{\sim}{\nu}}_{i}Q'QH_{i'} = QP^{\underset{\sim}{\nu}}_{i}H_{i'}$$

$$= Q\left\{\left(\sum_{j=0}^{\nu-p}\left(\frac{\lambda_{ij}}{b_j}\right)A^{\underset{\sim}{\nu}}_{j}\right)\left(\left(\bigotimes_{\ell=1}^{s_1}L_{i'\ell}\right)\otimes\left(\underset{\sim}{1}_{d_3}\right)\otimes \underset{\sim}{1}_{d_2}\right)\right\}$$

$$= d_2Q\left\{\left(\sum_{j=0}^{\nu-p}\frac{\lambda_{ij}}{b_j}\bigotimes_{\ell=1}^{s_1}M_{j\ell}L_{i'\ell}\right)\otimes\left(\bigotimes_{\ell=s_1+1}^{s_1+s_2}\underset{\sim}{1}_{a_\ell}^{i_\ell}\right)\otimes \underset{\sim}{1}_{d_2}\right\}, \tag{3.350}$$

$(i,i'=0,\cdots,\nu\text{-}p;\ \underset{\sim}{\nu}\in\Lambda)$. Define the matrix $N_{ii'}$ as

$$N_{ii'}=\sum_{j=0}^{\nu-p}\frac{\lambda_{ij}}{b_j}\bigotimes_{\ell=1}^{s_1}M_{j\ell}L_{i'\ell},\quad i,i'=0,1,\cdots,\nu\text{-}p. \tag{3.351}$$

Then, from (3.349) we see that for $i=0,1,\cdots,\nu\text{-}p$,

$$\begin{aligned}P_{\omega i}H_\omega&=P_{\omega i}(QH_0:QH_1:\cdots:QH_{\nu-p}) &&\Big(\text{see } (3.313)\Big)\\ &=P_{\omega i}(H_{\omega 0}:H_{\omega 1}:\cdots:H_{\omega,\nu-p}) &&\Big(\text{see } (3.336)\Big)\\ &=(d_1Q[N_{i0}\otimes\underset{\sim}{1}_{d_1}]:\cdots:d_1Q[N_{i,\nu-p}\otimes\underset{\sim}{1}_{d_1}])\\ &=d_1Q(N_i\otimes\underset{\sim}{1}_{d_1}), && (3.352)\end{aligned}$$

where

$$N_i=(N_{i0}:N_{i1}:\cdots:N_{i,\nu-p}),\quad i=0,1,\cdots,\nu\text{-}p. \tag{3.353}$$

Similarly, from (3.350) we have for $i=0,1,\cdots,\nu\text{-}p;\ \underset{\sim}{\nu}\in\Lambda$,

$$P^{\underset{\sim}{\nu}}_{\omega i}H_\omega=d_2Q\left(N_i\otimes\left(\bigotimes_{\ell=s_1+1}^{s_1+s_2}\underset{\sim}{1}^{i_\ell}_{a_\ell}\right)\otimes\underset{\sim}{1}_{d_2}\right). \tag{3.354}$$

Thus, for $i=0,1,\cdots,\nu\text{-}p;\ \underset{\sim}{\nu}\in\Lambda$,

$$\begin{aligned}\operatorname{rank}(P^{\underset{\sim}{\nu}}_{\omega i}H_\omega)&=\operatorname{rank}(N_i)=\operatorname{rank}(P_{\omega i}H_\omega)\\ &=\operatorname{rank}(QP_iQ'QH) &&\Big(\text{see } (3.346) \text{ and } (3.313)\Big)\end{aligned}$$

$$= \text{rank}(P_iH)$$

$$= \text{rank}(Q_i'Q_iH) \qquad \Big(\text{recall } (3.254)\Big)$$

$$= \text{rank}(Q_iH) \qquad (Q_i' \text{ has full column rank})$$

$$= m_i \qquad \Big(\text{by Lemma } 3.10(a)\Big). \tag{3.355}$$

Now, there exists a matrix C_i^* of order $m_i \times d_0$ $(d_0 = \prod_{i=1}^{s_1} a_i)$ and rank m_i such that $C_i^*N_i$ consists of m_i linearly independent rows of N_i, $i = 0,1,\cdots,\nu\text{-}p$. Define the $m_i \times c$ matrix $T_i^{\underset{\sim}{\nu}}$ as

$$T^{\underset{\sim}{\nu}}_{\omega i} = K^{\underset{\sim}{\nu}}_{\omega i} P^{\underset{\sim}{\nu}}_{\omega i} \tag{3.356}$$

where

$$K^{\underset{\sim}{\nu}}_{\omega i} = \left(C_i^* \otimes \Big(\bigotimes_{\ell=s_1+1}^{s_1+s_2} \underset{\sim}{1}_{a_\ell}^{i_\ell'}\Big) \otimes \underset{\sim}{1}_{d_2}'\right)Q', \tag{3.357}$$

$(i = 0,1,\cdots,\nu\text{-}p;\ \underset{\sim}{\nu} \in \Lambda)$.

The fixed-effects parameters: The sequence of estimators. For each $i = 0,1,\cdots,\nu\text{-}p$; $\underset{\sim}{\nu} \in \Lambda$ define the $c \times 1$ random vectors, $\underset{\sim}{z}_i^{\underset{\sim}{\nu}}$, as

$$\underset{\sim}{z}_i^{\underset{\sim}{\nu}} = T^{\underset{\sim}{\nu}}_{\omega i}\underset{\sim}{\omega}. \tag{3.358}$$

We now establish some distributional properties for the $\underset{\sim}{z}_i^{\underset{\sim}{\nu}}$. Before we do so, however, we will need the following lemma.

Lemma 3.12

Recall the definitions of P_i in (3.344) and $P_i^{\underset{\sim}{\nu}}$ in (3.347), $i = 0,1,\cdots,\nu\text{-}p;\ \underset{\sim}{\nu} \in \Lambda$. Then

(a)
$$P_i^{\underset{\sim}{\nu}} P_{i'}^{\underset{\sim}{\nu}} = \begin{cases} \frac{1}{d_3} P_i^{\underset{\sim}{\nu}}, & i = i' = 0,1,\cdots,\nu\text{-}p;\ \underset{\sim}{\nu} \in \Lambda \\ 0, & i \neq i';\ \underset{\sim}{\nu} \in \Lambda. \end{cases}$$

(b)
$$P_i P_{i'}^{\underset{\sim}{\nu}} = \begin{cases} \frac{1}{d_1}[I_{d_0} \otimes J_{d_1}] P_i^{\underset{\sim}{\nu}}, & i = i' = 0,1,\cdots,\nu\text{-}p;\ \underset{\sim}{\nu} \in \Lambda \\ 0, & i \neq i';\ \underset{\sim}{\nu} \in \Lambda. \end{cases}$$

Proof:

(a) From the fact that P_i is idempotent, $i = 1,\cdots,\nu$ (see Khuri's (1990) Lemma 3.1), we have

$$P_i P_i = \Big(\sum_{j=0}^{\nu-p} \frac{\lambda_{ij}}{b_j} A_j \Big)^2 = \sum_{j=0}^{\nu-p} \frac{\lambda_{ij}^2}{b_j^2} A_j A_j + \sum_{j \neq j'} \frac{\lambda_{ij}\lambda_{ij'}}{b_j b_{j'}} A_j A_{j'}$$

$$= d_1 \sum_{j=0}^{\nu-p} \frac{\lambda_{ij}^2}{b_j^2} \Big\{ \bigotimes_{\ell=1}^{s_1} M_{j\ell} M_{j\ell} \otimes J_{d_1} \Big\}$$

$$+ d_1 \sum_{j \neq j'} \frac{\lambda_{ij}\lambda_{ij'}}{b_j b_{j'}} \Big\{ \bigotimes_{\ell=1}^{s_1} M_{j\ell} M_{j'\ell} \otimes J_{d_1} \Big\} \qquad \text{(see (3.340))}$$

$$= d_1 \left\{ \left[\Big(\sum_{j=0}^{\nu-p} \frac{\lambda_{ij}^2}{b_j^2} \bigotimes_{\ell=1}^{s_1} M_{j\ell} M_{j\ell} \Big) \otimes J_{d_1} \right] \right.$$

$$+\left[\left(\sum_{j\neq j'}\frac{\lambda_{ij}\lambda_{ij'}}{b_j b_{j'}}\bigotimes_{\ell=1}^{s_1} M_{j\ell}M_{j'\ell}\right)\otimes J_{d_1}\right]\Bigg\}$$

$$=d_1\left\{\left(\sum_{j=0}^{\nu-p}\frac{\lambda_{ij}}{b_j}\bigotimes_{\ell=1}^{s_1} M_{j\ell}\right)^2\otimes J_{d_1}\right\}$$

$$=\left(\sum_{j=0}^{\nu-p}\frac{\lambda_{ij}}{b_j}\bigotimes_{\ell=1}^{s_1} M_{j\ell}\right)\otimes J_{d_1}=P_i,\quad i=0,1,\cdots,\nu\text{-}p.$$

Consequently, we must have

$$d_1\left(\sum_{j=0}^{\nu-p}\frac{\lambda_{ij}}{b_j}\bigotimes_{\ell=1}^{s_1} M_{j\ell}\right)^2=\sum_{j=0}^{\nu-p}\frac{\lambda_{ij}}{b_j}\bigotimes_{\ell=1}^{s_1} M_{j\ell}. \tag{3.359}$$

Thus, for $i=0,1,\cdots,\nu\text{-}p$; $\underset{\sim}{\nu}\in\Lambda$, we obtain

$$P_i^{\underset{\sim}{\nu}}P_i^{\underset{\sim}{\nu}}=\left(\sum_{j=0}^{\nu-p}\frac{\lambda_{ij}}{b_j}A_i^{\underset{\sim}{\nu}}\right)^2 \qquad \left(\text{see } (3.347)\right)$$

$$=\left(\sum_{j=0}^{\nu-p}\frac{\lambda_{ij}}{b_j}\left[\bigotimes_{\ell=1}^{s_1} M_{j\ell}\otimes\left(\bigotimes_{\ell=s_1+1}^{s_1+s_2} J_{a_\ell}^{i_\ell}\right)\otimes J_{d_2}\right]\right)^2 \qquad \left(\text{see } (3.342)\right)$$

$$=\left[\left(\sum_{j=0}^{\nu-p}\frac{\lambda_{ij}}{b_j}\bigotimes_{\ell=1}^{s_1} M_{j\ell}\right)\otimes\left(\bigotimes_{\ell=s_1+1}^{s_1+s_2} J_{a_\ell}^{i_\ell}\right)\otimes J_{d_2}\right]^2$$

$$=d_2\left[\left(\sum_{j=0}^{\nu-p}\frac{\lambda_{ij}}{b_j}\bigotimes_{\ell=1}^{s_1} M_{j\ell}\right)^2\otimes\left(\bigotimes_{\ell=s_1+1}^{s_1+s_2} J_{a_\ell}^{i_\ell}\right)\otimes J_{d_2}\right]$$

$$= d_2\left[\left(\frac{1}{d_1}\sum_{j=0}^{\nu-p}\frac{\lambda_{ij}}{b_j}\bigotimes_{\ell=1}^{s_1} M_{j\ell}\right)\otimes\left(\bigotimes_{\ell=s_1+1}^{s_1+s_2} J_{a_\ell}^{i_\ell}\right)\otimes J_{d_2}\right] \qquad \Big(\text{from } (3.359)\Big)$$

$$= \frac{1}{d_3}P_{i}^{\nu} \qquad \Big(\text{see } (3.331)\Big).$$

Since $P_iP_{i'} = 0$ for $i \neq i'$ (see Khuri's (1990) Lemma 3.1) we have, for $i \neq i'$,

$$0 = P_iP_{i'} = \left(\sum_{j=0}^{\nu-p}\frac{\lambda_{ij}}{b_j}A_j\right)\left(\sum_{j'=0}^{\nu-p}\frac{\lambda_{i'j'}}{b_{j'}}A_{j'}\right)$$

$$= \sum_{j=0}^{\nu-p}\frac{\lambda_{ij}\lambda_{i'j}}{b_j^2}A_jA_j + \sum_{j\neq j'}\frac{\lambda_{ij}\lambda_{i'j'}}{b_jb_{j'}}A_jA_{j'}.$$

After some simplification similar to the above we obtain

$$0 = d_1\left\{\left[\left(\sum_{j=0}^{\nu-p}\frac{\lambda_{ij}\lambda_{i'j}}{b_j^2}\bigotimes_{\ell=1}^{s_1}M_{j\ell}M_{j\ell}\right)\otimes J_{d_1}\right]\right.$$

$$\left.+\left[\left(\sum_{j\neq j'}\frac{\lambda_{ij}\lambda_{i'j'}}{b_jb_{j'}}\bigotimes_{\ell=1}^{s_1}M_{j\ell}M_{j'\ell}\right)\otimes J_{d_1}\right]\right\}$$

$$= d_1\left\{\left[\left(\sum_{j=0}^{\nu-p}\frac{\lambda_{ij}}{b_j}\bigotimes_{\ell=1}^{s_1}M_{j\ell}\right)\left(\sum_{j'=0}^{\nu-p}\frac{\lambda_{i'j'}}{b_{j'}}M_{j'\ell}\right)\right]\otimes J_{d_1}\right\}$$

$$\Rightarrow \sum_{j=0}^{\nu-p}\sum_{j'=0}^{\nu-p}\frac{\lambda_{ij}\lambda_{i'j'}}{b_jb_{j'}}\bigotimes_{\ell=1}^{s_1}M_{j\ell}M_{j'\ell} = 0. \qquad (3.360)$$

Thus, for $i \neq i'$; $\underset{\sim}{\nu} \in \Lambda$, we have

$$P_{i}^{\underset{\sim}{\nu}} P_{i'}^{\underset{\sim}{\nu}} = \sum_{j=0}^{\nu-p} \sum_{j'=0}^{\nu-p} \frac{\lambda_{ij}\lambda_{i'j'}}{b_j b_{j'}} \; A_j^{\underset{\sim}{\nu}} A_{j'}^{\underset{\sim}{\nu}}$$

$$= d_2 \sum_{j=0}^{\nu-p} \sum_{j'=0}^{\nu-p} \frac{\lambda_{ij}\lambda_{i'j'}}{b_j b_{j'}} \left\{ \bigotimes_{\ell=1}^{s_1} M_{j\ell} M_{j'\ell} \otimes \left(\bigotimes_{\ell=s_1+1}^{s_1+s_2} J_{a_\ell}^{i_\ell} \right) \otimes J_{d_2} \right\} \quad \Big(\text{see } (3.342)\Big)$$

$$= d_2 \left\{ \left(\sum_{j=0}^{\nu-p} \sum_{j'=0}^{\nu-p} \frac{\lambda_{ij}\lambda_{i'j'}}{b_j b_{j'}} \bigotimes_{\ell=1}^{s_1} M_{j\ell} M_{j'\ell} \right) \otimes \left(\bigotimes_{\ell=s_1+1}^{s_1+s_2} J_{a_\ell}^{i_\ell} \right) \otimes J_{d_2} \right\}$$

$$= 0 \qquad \Big(\text{by } (3.360)\Big).$$

(b) For all $i = i' = 0, 1, \cdots, \nu-p$ and $\underset{\sim}{\nu} \in \Lambda$,

$$P_i P_i^{\underset{\sim}{\nu}} = \left(\sum_{j=0}^{\nu-p} \frac{\lambda_{ij}}{b_j} \; A_j \right) \left(\sum_{j'=0}^{\nu-p} \frac{\lambda_{ij'}}{b_{j'}} \; A_{j'}^{\underset{\sim}{\nu}} \right)$$

$$= \sum_{j=0}^{\nu-p} \frac{\lambda_{ij}^2}{b_j^2} \; A_j A_j^{\underset{\sim}{\nu}} + \sum_{j \neq j'} \frac{\lambda_{ij}\lambda_{ij'}}{b_j b_{j'}} A_j A_{j'}^{\underset{\sim}{\nu}}.$$

From (3.340) and (3.342) we can express this as

$$= d_2 \sum_{j=0}^{\nu-p} \frac{\lambda_{ij}^2}{b_j^2} \left[\bigotimes_{\ell=1}^{s_1} M_{j\ell} M_{j\ell} \otimes J_{d_3} \left(\bigotimes_{\ell=s_1+1}^{s_1+s_2} J_{a_\ell}^{i_\ell} \right) \otimes J_{d_2} \right]$$

$$+ d_2 \sum_{j \neq j'} \frac{\lambda_{ij}\lambda_{ij'}}{b_j b_{j'}} \left[\bigotimes_{\ell=1}^{s_1} M_{j\ell} M_{j'\ell} \otimes J_{d_3} \left(\bigotimes_{\ell=s_1+1}^{s_1+s_2} J_{a_\ell}^{i_\ell} \right) \otimes J_{d_2} \right]$$

$$= d_2 \Bigg\{ \left[\sum_{j=0}^{\nu-p} \frac{\lambda_{ij}^2}{b_j^2} \bigotimes_{\ell=1}^{s_1} M_{j\ell} M_{j\ell} + \sum_{j \neq j'} \frac{\lambda_{ij}\lambda_{ij'}}{b_j b_{j'}} \bigotimes_{\ell=1}^{s_1} M_{j\ell} M_{j'\ell} \right]$$

$$\otimes J_{d_3}\Big(\bigotimes_{\ell=s_1+1}^{s_1+s_2} J_{a_\ell}^{i_\ell}\Big)\otimes J_{d_2}\Big\}$$

$$= d_2\left\{\left[\sum_{j=0}^{\nu-p}\frac{\lambda_{ij}}{b_j}\bigotimes_{\ell=1}^{s_1} M_{j\ell}\right]^2 \otimes J_{d_3}\Big(\bigotimes_{\ell=s_1+1}^{s_1+s_2} J_{a_\ell}^{i_\ell}\Big)\otimes J_{d_2}\right\}$$

$$= \frac{1}{d_3}\left\{\left[\sum_{j=0}^{\nu-p}\frac{\lambda_{ij}}{b_j}\bigotimes_{\ell=1}^{s_1} M_{j\ell}\right] \otimes J_{d_3}\Big(\bigotimes_{\ell=s_1+1}^{s_1+s_2} J_{a_\ell}^{i_\ell}\Big)\otimes J_{d_2}\right\} \qquad \Big(\text{see } (3.359)\Big)$$

$$= \frac{1}{d_1}\left\{\left[I_{d_0}\otimes J_{d_3}\otimes J_{d_2}\right]\left[\Big(\sum_{j=0}^{\nu-p}\frac{\lambda_{ij}}{b_j}\bigotimes_{\ell=1}^{s_1} M_{j\ell}\Big)\otimes\Big(\bigotimes_{\ell=s_1+1}^{s_1+s_2} J_{a_\ell}^{i_\ell}\Big)\otimes J_{d_2}\right]\right\}$$

$$= \frac{1}{d_1}\left[I_{d_0}\otimes J_{d_1}\right]P_{\underset{\sim}{i}}^{\underset{\sim}{\nu}}.$$

When $i \neq i'$ and $\underset{\sim}{\nu}\in\Lambda$ we have, reasoning similarly to the above and using (3.360), that

$$P_{\underset{\sim}{i}}P_{\underset{\sim}{i}'}^{\underset{\sim}{\nu}} = 0. \qquad \square$$

Now, recalling (3.358) and using Lemma 3.12, we have for $i = 0,1,\cdots,\nu-p$; $\underset{\sim}{\nu}\in\Lambda$,

(i) $$E(\underset{\sim}{z}_{\underset{\sim}{i}}^{\underset{\sim}{\nu}}) = T_{\omega\underset{\sim}{i}}^{\underset{\sim}{\nu}}E(\underset{\sim}{\omega}) = T_{\omega\underset{\sim}{i}}^{\underset{\sim}{\nu}}H_\omega\underset{\sim}{g} \qquad \Big(\text{see } (3.317)\Big)$$

$$= K_{\omega\underset{\sim}{i}}^{\underset{\sim}{\nu}}P_{\omega\underset{\sim}{i}}^{\underset{\sim}{\nu}}H_\omega\underset{\sim}{g} \qquad \Big(\text{see } (3.356)\Big)$$

$$= K_{\omega\underset{\sim}{i}}^{\underset{\sim}{\nu}}\left[d_2Q\Big(N_{\underset{\sim}{i}}\otimes\Big(\bigotimes_{\ell=s_1+1}^{s_1+s_2}\underset{\sim}{1}_{a_\ell}^{i_\ell}\Big)\otimes\underset{\sim}{1}_{d_2}\Big)\right]\underset{\sim}{g} \qquad \Big(\text{see } (3.354)\Big)$$

$$= d_2\left[C_i^* N_i \otimes (1) \otimes d_2\right] \underset{\sim}{g} \qquad \left(\text{see } (3.357)\right)$$

$$= d_2^2 C_i^* N_i \underset{\sim}{g}. \tag{3.361}$$

(ii) $\mathrm{Var}(\underset{\sim}{z}_i^{\nu}) = T^{\nu}_{\underset{\sim}{\omega} i}\left[\mathrm{Var}(\underset{\sim}{\omega})\right] T^{\nu\prime}_{\underset{\sim}{\omega} i}$

$$= K^{\nu}_{\underset{\sim}{\omega} i} P^{\nu}_{\underset{\sim}{\omega} i}\left[F_{\omega}\Big(\bigoplus_{i=\nu-p+1}^{\nu} \sigma_i^2\Big) F'_{\omega} + \frac{1}{n^{(1)}}\sigma_\epsilon^2 I_c\right] P^{\nu\prime}_{\underset{\sim}{\omega} i} K^{\nu\prime}_{\underset{\sim}{\omega} i}.$$

The term in the brackets follows from (3.316) and the facts that $\mathrm{Var}(\underset{\sim}{\epsilon}_{\omega}) = \frac{1}{n^{(1)}}\sigma_\epsilon^2 I_c$ $\left(\text{see } (3.315)\right)$ and $\underset{\sim}{h} = (\underset{\sim}{\beta}'_{\nu-p+1} : \cdots : \underset{\sim}{\beta}'_{\nu})'$ with $\mathrm{Var}(\underset{\sim}{\beta}_i) = \sigma_i^2 I_{c_i}$, $i = \nu-p+1, \cdots, \nu$ $\left(\text{see } (3.196)\right)$. Now from (3.314) and (3.348) we have, since $P^{\underset{\sim}{\nu}}_i$ is symmetric $\big($see (3.347) and (3.342)$\big)$,

$$\mathrm{Var}(\underset{\sim}{z}_i^{\nu}) = K^{\nu}_{\underset{\sim}{\omega} i} Q\left[P^{\underset{\sim}{\nu}}_i F\Big(\bigoplus_{i=\nu-p+1}^{\nu} \sigma_i^2\Big) F' P^{\underset{\sim}{\nu}}_i + \frac{1}{n^{(1)}}\sigma_\epsilon^2 P^{\underset{\sim}{\nu}}_i P^{\underset{\sim}{\nu}}_i\right] Q' K^{\nu\prime}_{\underset{\sim}{\omega} i}.$$

Now, the first term in the bracket can be written as

$$P^{\underset{\sim}{\nu}}_i F\Big(\bigoplus_{i=\nu-p+1}^{\nu} \sigma_i^2\Big) F' P^{\underset{\sim}{\nu}}_i = P^{\underset{\sim}{\nu}}_i\Big(\sum_{j=\nu-p+1}^{\nu} \sigma_j^2 A_j\Big) P^{\underset{\sim}{\nu}}_i \qquad \left(\text{see } (3.210)\right)$$

$$= \sum_{j=\nu-p+1}^{\nu} \sigma_j^2 \kappa_{ij}^* P^{\underset{\sim}{\nu}}_i \qquad \left(\text{see Khuri's (1984) Lemma 5.2}\right)$$

where

$$\kappa_{ij}^* = \begin{cases} 0, & \psi_i \not\subset \psi_j \\ b_j^*/d_3, & \psi_i \subset \psi_j, \end{cases} \tag{3.362}$$

$(i = 0, 1, \cdots, \nu-p; j = \nu-p+1, \cdots, \nu;\ \underset{\sim}{\nu} \in \Lambda)$. Also

$$b_j^* = \frac{b_j}{\prod_{\ell=s_1+1}^{s_1+s_2} f_{j\ell}^c}, \quad j=\nu-p+1,\cdots,\nu \tag{3.363}$$

where

$$f_{j\ell}^c = \begin{cases} a_\ell, & k_\ell \in \psi_j^c \\ 1, & k_\ell \in \psi_j, \end{cases} \tag{3.364}$$

(see Khuri (1984)). Therefore, recalling (3.217),

$$\sum_{j=\nu-p+1}^{\nu} \sigma_j^2 \kappa_{ij}^* P_i^{\underset{\sim}{\nu}} = \frac{1}{d_3}\Big(\sum_{j\in W_i} b_j^* \sigma_j^2\Big) P_i^{\underset{\sim}{\nu}}.$$

Hence, from Lemma 3.12(a) and (3.357),

$$\begin{aligned} \mathrm{Var}(\underset{\sim}{z}_i^{\underset{\sim}{\nu}}) &= K_{\omega i}^{\underset{\sim}{\nu}} Q\left[\frac{1}{d_3}\Big(\sum_{j\in W_i} b_j^*\sigma_j^2\Big)P_i^{\underset{\sim}{\nu}} + \frac{1}{d_3}\,\frac{1}{n^{(1)}}\sigma_\epsilon^2 P_i^{\underset{\sim}{\nu}}\right] Q' K_{\omega i}^{\underset{\sim}{\nu}\prime} \\ &= \frac{1}{d_3}\Bigg\{\Big(C_i^* \otimes \Big(\bigotimes_{\ell=s_1+1}^{s_1+s_2} \underset{\sim}{1}_{a_\ell}^{i'_\ell}\Big)\otimes \underset{\sim}{1}'_{d_2}\Big)\Big(\Big(\sum_{j\in W_i} b_j^*\sigma_j^2 + \frac{\sigma_\epsilon^2}{n^{(1)}}\Big)P_i^{\underset{\sim}{\nu}}\Big) \\ &\quad \times \Big(C_i^{*\prime} \otimes \Big(\bigotimes_{\ell=s_1+1}^{s_1+s_2} \underset{\sim}{1}_{a_\ell}^{i_\ell}\Big)\otimes \underset{\sim}{1}_{d_2}\Big)\Bigg\}. \end{aligned} \tag{3.365}$$

Now, let

$$B_i = \sum_{j=0}^{\nu-p} \frac{\lambda_{ij}}{b_j} \bigotimes_{\ell=1}^{s_1} M_{j\ell}, \quad i=0,1,\cdots,\nu-p \tag{3.366}$$

$$\xi_i^* = \frac{1}{d_3}\Big(\sum_{j\in W_i} b_j^*\sigma_j^2 + \sigma_\epsilon^2/n^{(1)}\Big), \quad i=0,1,\cdots,\nu-p, \tag{3.367}$$

then noting that $P_i^{\underset{\sim}{\nu}} = B_i \otimes \left(\bigotimes_{\ell=s_1+1}^{s_1+s_2} J_{a_\ell}^{i_\ell} \right) \otimes J_{d_2}$, $i = 0,1,\cdots,\nu\text{-}p$ (see (3.347) and (3.342)), we have that (3.365) can be expressed as

$$\xi_i^* \left[C_i^* B_i C_i^{*\prime} \otimes (1) \otimes (d_2^2) \right] = \xi_i^* d_2^2 C_i^* B_i C_i^{*\prime}, \tag{3.368}$$

$(i = 0,1,\cdots,\nu\text{-}p;\ \underset{\sim}{\nu} \in \Lambda)$.

(iii) For $i \neq i'$, $\underset{\sim}{\nu} \in \Lambda$, and recalling $P_{\underset{\sim}{\omega} i}^{\underset{\sim}{\nu}}$ is symmetric

$$\operatorname{Cov}(\underset{\sim}{z}_i^{\underset{\sim}{\nu}}, \underset{\sim}{z}_{i'}^{\underset{\sim}{\nu}}) = T_{\underset{\sim}{\omega} i}^{\underset{\sim}{\nu}} [\operatorname{Var}(\underset{\sim}{\omega})] T_{\underset{\sim}{\omega} i'}^{\underset{\sim}{\nu}\prime}, \qquad \left(\text{see } (3.358)\right)$$

$$= K_{\underset{\sim}{\omega} i}^{\underset{\sim}{\nu}} P_{\underset{\sim}{\omega} i}^{\underset{\sim}{\nu}} \left[\bigoplus_{i=0}^{\nu} \xi_i I_{m_i} \right] P_{\underset{\sim}{\omega} i'}^{\underset{\sim}{\nu}} K_{\underset{\sim}{\omega} i'}^{\underset{\sim}{\nu}\prime} \qquad \left(\text{see } (3.356) \text{ and } (3.317)\right)$$

$$= K_{\underset{\sim}{\omega} i}^{\underset{\sim}{\nu}} Q P_i^{\underset{\sim}{\nu}} [Q_0' : \cdots : Q_\nu'] \left(\bigoplus_{i=0}^{\nu} \xi_i I_{m_i} \right) \times \begin{bmatrix} Q_0 \\ \vdots \\ Q_\nu \end{bmatrix} P_{i'}^{\underset{\sim}{\nu}} Q' K_{\underset{\sim}{\omega} i'}^{\underset{\sim}{\nu}\prime} \qquad \left(\text{see } (3.348) \text{ and } (3.214)\right)$$

$$= K_{\underset{\sim}{\omega} i}^{\underset{\sim}{\nu}} Q P_i^{\underset{\sim}{\nu}} \left(\sum_{j=0}^{\nu} \xi_j Q_j' Q_j \right) P_{i'}^{\underset{\sim}{\nu}} Q' K_{\underset{\sim}{\omega} i'}^{\underset{\sim}{\nu}\prime}$$

$$= K_{\underset{\sim}{\omega} i}^{\underset{\sim}{\nu}} Q P_i^{\underset{\sim}{\nu}} \left(\sum_{j=0}^{\nu} \xi_j P_j \right) P_{i'}^{\underset{\sim}{\nu}} Q' K_{\underset{\sim}{\omega} i'}^{\underset{\sim}{\nu}\prime} \qquad \left(\text{see } (3.252)\right)$$

$$= K_{\underset{\sim}{\omega} i}^{\underset{\sim}{\nu}} Q P_i^{\underset{\sim}{\nu}} \left[\frac{\xi_{i'}}{d_1} P_{i'} \left(I_{d_0} \otimes J_{d_1} \right) P_{i'}^{\underset{\sim}{\nu}} \right] Q' K_{\underset{\sim}{\omega} i'}^{\underset{\sim}{\nu}\prime} = 0 \tag{3.369}$$

This follows from Lemma 3.12(b) and the fact that P_i is idempotent for $i = 0,1,\cdots,\nu\text{-}p$. We have therefore just proven the following theorem.

Theorem 3.11

The random vectors $\underset{\sim}{z}_0^{\nu}, \underset{\sim}{z}_1^{\nu}, \cdots, \underset{\sim}{z}_{\nu-p}^{\nu}$ are independent normally distributed vectors with

(a) $E(\underset{\sim}{z}_i^{\nu}) = d_2^2 C_i^* N_i \underset{\sim}{g}, \ i = 0,1,\cdots,\nu\text{-}p; \ \underset{\sim}{\nu} \in \Lambda$

(b) $\mathrm{Var}(\underset{\sim}{z}_i^{\nu}) = \xi_i^* d_2^2 C_i^* B_i C_i^{*\prime}, \ i = 0,1,\cdots,\nu\text{-}p; \ \underset{\sim}{\nu} \in \Lambda$

where d_2 was defined in (3.330); $\underset{\sim}{g}$ is the vector of fixed effects defined in (3.196); C_i^* is such that $C_i^* N_i$ consist of m_i linearly independent rows of N_i $\big(N_i$ was given in (3.353)$\big)$; B_i was given in (3.366); and ξ_i^* was defined in (3.367).

This theorem shows that the random vectors in the set $\{\underset{\sim}{z}_j^{\nu} : \underset{\sim}{\nu} \in \Lambda\}$ share the same mean vector, $d_2^2 C_j^* N_j \underset{\sim}{g}$, and the same variance-covariance matrix, $\xi_j^* d_2^2 C_j^* B_j C_j^{*\prime}$, $j = 0,1,\cdots,\nu\text{-}p$. The matrix $C_j^* B_j C_j^{*\prime}$ $(j = 0,1,\cdots,\nu\text{-}p)$ is nonsingular since if it were singular, the rows of $T_{\omega j}^{\underset{\sim}{\nu}}$ $\big($see (3.356) and (3.358)$\big)$ would be linearly dependent. But this implies that the rows of $T_{\omega j}^{\underset{\sim}{\nu}} H_{\omega}$ would be linearly dependent which is a contradiction since $T_{\omega j}^{\underset{\sim}{\nu}} H_{\omega} = d_2^2 C_j^* N_j$ by (3.361) and C_j^* was determined so that $C_j^* N_j$ would have linearly independent rows.

Further, $C_j^* N_j \underset{\sim}{g}$ is estimable for all $j = 0,1,\cdots,\nu\text{-}p$ since

$$d_2^2 C_j^* N_j \underset{\sim}{g} = T_{\omega j}^{\underset{\sim}{\nu}} H_{\omega} \underset{\sim}{g} \qquad \big(\text{by } (3.361)\big)$$

$$= T_{\omega j}^{\underset{\sim}{\nu}} Q H \underset{\sim}{g} \qquad \big(\text{by } (3.313)\big)$$

$$=T_{\omega j}^{\underline{\nu}}QE(\underline{y}) \qquad \big(\text{by } (3.210)\big).$$

Also, for each $j=0,1,\cdots,\nu\text{-}p$ and $\underline{\nu}\in\Lambda$, the rows of $P_{\omega j}^{\underline{\nu}}H_{\omega}$ and those of $P_{\omega j}H_{\omega}$ span the same vector space spanned by the rows of N_j and hence by $C_j^*N_j$ $\big($see (3.354) and (3.355)$\big)$. Since $P_{\omega j}H_{\omega}=QP_jQ'QH=QP_jH$ $\big($see (3.346) and (3.313)$\big)$, the space spanned by $\{QP_0H\underline{g},\cdots,QP_{\nu-p}H\underline{g}\}$ which is the same as the space spanned by $\{Q_0'Q_0H\underline{g},\cdots,Q_{\nu-p}'Q_{\nu-p}H\underline{g}\}$ is the space of estimable linear functions of the fixed-effects parameters in model (3.316) $\big($recall Lemma 3.10(c) and (3.352)$\big)$. Thus $\{C_0^*N_0\underline{g},\cdots,C_{\nu-p}^*N_{\nu-p}\underline{g}\}$ must also span the space of estimable linear functions of the fixed effects parameters. Consequently, we can restrict our attention to the development of exact procedures concerning $C_0^*N_0\underline{g},\cdots,C_{\nu-p}^*N_{\nu-p}\underline{g}$.

The fixed-effects parameters: Exact procedures. Consider the set

$$\Lambda_0=\{\underline{\nu}:\underline{\nu}=q\underline{1}_{s_2},\ 1\le q\le\tau_0\} \tag{3.370}$$

where q is constant and

$$\tau_0=\min_{s_1+1\le i\le s_1+s_2}a_i. \tag{3.371}$$

Also, define the subset of vectors $\mathcal{J}_j$ to be

$$\mathcal{J}_j=\{\underline{z}_j^{\underline{\nu}}:\underline{\nu}\in\Lambda_0\},\quad j=0,1,\cdots,\nu\text{-}p. \tag{3.372}$$

Then as explained in Khuri (1984, p.21) the sequence of τ_0 random vectors in $\mathcal{J}_j$ $(j=0,1,\cdots,\nu\text{-}p)$ are statistically independent. Since these random vectors are normally distributed with the same mean vector and variance-covariance matrix (see Theorem 3.11), it follows that an unbiased estimator of ξ_j^* (see (3.367) and Theorem 3.11(b)) is given by

$$\mathrm{tr}[S_j(d_2^2C_j^*B_jC_j^{*\prime})^{-1}/m_j(\tau_0-1)] \tag{3.373}$$

where S_j is the matrix

$$S_j = \sum_{\underset{\sim}{\nu} \in \Lambda_0} (\underset{\sim}{z}_j^{\underset{\sim}{\nu}} - \overline{\underset{\sim}{z}}_j)(\underset{\sim}{z}_j^{\underset{\sim}{\nu}} - \overline{\underset{\sim}{z}}_j)' \tag{3.374}$$

and

$$\overline{\underset{\sim}{z}}_j = \frac{1}{\tau_0} \sum_{\underset{\sim}{\nu} \in \Lambda_0} \underset{\sim}{z}_j^{\underset{\sim}{\nu}}, \tag{3.375}$$

$j=0,1,\cdots,\nu\text{-}p$ (see Anderson (1984, p. 430)). Further,

$$\mathrm{tr}[S_j(d_2^2C_j^*B_jC_j^{*\prime})^{-1}]/\xi_j^* \sim \chi^2_{m_j(\tau_0-1)}, \tag{3.376}$$

$(j=0,1,\cdots,\nu\text{-}p)$. Since

$$\overline{\underset{\sim}{z}}_j \sim N(d_2^2C_j^*N_j\underset{\sim}{g},\ \xi_j^*d_2^2C_j^*B_jC_j^{*\prime}/\tau_0),$$

$(j=0,1,\cdots,\nu\text{-}p)$, it follows that

$$\tau_0(\overline{\underset{\sim}{z}}_j - d_2^2C_j^*N_j\underset{\sim}{g})'(d_2^2C_j^*B_jC_j^*)^{-1}(\overline{\underset{\sim}{z}}_j - d_2^2C_j^*N_j\underset{\sim}{g})/\xi_j^* \sim \chi^2_{m_j}, \tag{3.377}$$

$(j=0,1,\cdots,\nu\text{-}p)$. Finally, because $\overline{\underset{\sim}{z}}_j$ and S_j are independent (see Anderson (1984, p. 71)) we conclude that

$$\tau_0(\tau_0-1)\frac{(\underset{\sim}{\bar{z}}_j - d_2^2C_j^*N_j\underset{\sim}{g})'(d_2^2C_j^*B_jC_j^{*\prime})^{-1}(\underset{\sim}{\bar{z}}_j - d_2^2C_j^*N_j\underset{\sim}{g})}{tr[S_j(d_2^2C_j^*B_jC_j^{*\prime})^{-1}]} \sim F_{m_j,m_j(\tau_0-1)}, \qquad (3.378)$$

$(j=0,1,\cdots,\nu\text{-}p)$. This formula allows us to test a testable hypothesis concerning $C_j^*N_j\underset{\sim}{g}(j=0,1,\cdots,\nu\text{-}p)$ using an exact F-test.

Simultaneous confidence intervals on linear functions of $C_j^*N_j\underset{\sim}{g}$ $(j=0,1,\cdots,\nu\text{-}p)$ can be obtained from the formula

$$P\left\{d_2^2\underset{\sim}{\ell}'C_j^*N_j\underset{\sim}{g} \in \underset{\sim}{\ell}'\underset{\sim}{\bar{z}}_j \pm c_0[\underset{\sim}{\ell}'(d_2^2C_j^*B_jC_j^{*\prime})\underset{\sim}{\ell}]^{\frac{1}{2}}\right\} = 1-\alpha, \qquad (3.379)$$

$(j=0,1,\cdots,\nu\text{-}p)$, where $\underset{\sim}{\ell}$ is a vector of constant coefficients and

$$c_0 = \left\{tr[S_j(d_2^2C_j^*B_jC_j^{*\prime})^{-1}]F_{\alpha,m_j,m_j(\tau_0-1)}/\tau_0(\tau_0-1)\right\}^{\frac{1}{2}}.$$

As a last remark, the matrix C_j^* $(j=0,1,\cdots,\nu\text{-}p)$ is not unique. However, as pointed out in Khuri (1984, p. 22), the F-statistic given in (3.378) and hence the confidence interval given in (3.379) is invariant to the choice of the matrix C_j^*.

3.3.8. Numerical Examples

Dyeing of cotton-synthetic cloth. To begin with we use the data in Table 3.6 to illustrate the methodology developed in Sections 3.3 and 3.4 of this chapter. The data in this table represent numerical scores resulting from a comparison between a finished version of a dyed cotton-synthetic cloth and a standard. Two temperatures (Factor A), three cycle times (Factor B), and three operators (Factor C) were selected and at most three small specimens of cloth were dyed under each set of conditions (see Montgomery (1984, p. 245)). We assume that Factors A and B are fixed while Factor C is random. The appropriate model is given by

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \gamma_k + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \epsilon_{ijkl},$$

$i=1,2;\ j=1,2,3;\ k=1,2,3;\ \ell=1,2,\cdots,n_{ijk}$.

Our first task is to analyze the variance components. To aid in the understanding of the application of the testing procedures, the reader is referred to Table 3.7 which lists some key quantities used in the development of the exact tests. The expected mean square values of $MS_{ri} = SS_{ri}/m_i$, $i = 4,5,6,7$, are provided in Table 3.8. The results are summarized in Table 3.9.

We can see from Table 3.8 that no exact test (based on our procedure) exists for testing $H_0: \sigma^2_\gamma = 0$. Thus we must apply Satterthwaite's procedure and construct an approximate test. The test statistic in this case is given by

$$F_4 = \frac{MS_{r4}}{MS_{r5} + MS_{r6} - MS_{r7}} = 1.547$$

which has an approximate F-distribution with 2 and η^* degrees of freedom, where

$$\eta^* = \frac{(MS_{r5} + MS_{r6} - MS_{r7})^2}{\frac{(MS_{r5})^2}{2} + \frac{(MS_{r6})^2}{4} + \frac{(MS_{r7})^2}{4}} = 2.728.$$

The observed level of significance is 0.3356.

Table 3.6. Numerical scores resulting from a comparison between dyed cotton-synthetic cloth and a standard

	Temperature					
	300°			350°		
	Operator			Operator		
Cycle Time	1	2	3	1	2	3
40	23	28	31	24	36	36
	25	26	32		35	39
			29			
50	36	38	33	37	34	34
	35	39		35	38	
	36				36	
60	28	35	26	26	36	28
	24	35	27	29	37	26
		34	25		34	24

(source: Montgomery (1984, p. 245))

We now turn our attention to the analysis of the fixed-effects. The reader is referred to Table 3.10 which lists some quantities needed in the construction of the exact tests (see also Table 3.7). From Lemma 3.10(c) we may restrict our focus to testing hypothesis concerning $\underset{\sim}{\tau}_0,\underset{\sim}{\tau}_1,\underset{\sim}{\tau}_2$, and $\underset{\sim}{\tau}_3$. In this case these quantities become:

$$\underset{\sim}{\tau}_0 = (\mu + \bar{\alpha} + \bar{\beta} + (\overline{\alpha\beta}))\underset{\sim}{1}_{18}$$

$$\underset{\sim}{\tau}_1 = \begin{bmatrix} (\alpha_1 - \bar{\alpha} + (\overline{\alpha\beta})_{1.} - (\overline{\alpha\beta}))\underset{\sim}{1}_9 \\ (\alpha_2 - \bar{\alpha} + (\overline{\alpha\beta})_{2.} - (\overline{\alpha\beta}))\underset{\sim}{1}_9 \end{bmatrix}$$

$$\underset{\sim}{\tau}_2 = \begin{bmatrix} (\beta_1 - \bar{\beta} + (\overline{\alpha\beta})_{.1} - (\overline{\alpha\beta}))\underset{\sim}{1}_6 \\ (\beta_2 - \bar{\beta} + (\overline{\alpha\beta})_{.2} - (\overline{\alpha\beta}))\underset{\sim}{1}_6 \\ (\beta_3 - \bar{\beta} + (\overline{\alpha\beta})_{.3} - (\overline{\alpha\beta}))\underset{\sim}{1}_6 \end{bmatrix}$$

$$\underset{\sim}{\tau}_3 = \tfrac{1}{2} \begin{bmatrix} ((\alpha\beta)_{11} - (\overline{\alpha\beta})_{1.} - (\alpha\beta)_{21} + (\overline{\alpha\beta})_{2.})\underset{\sim}{1}_3 \\ ((\alpha\beta)_{12} - (\overline{\alpha\beta})_{1.} - (\alpha\beta)_{22} + (\overline{\alpha\beta})_{2.})\underset{\sim}{1}_3 \\ ((\alpha\beta)_{13} - (\overline{\alpha\beta})_{1.} - (\alpha\beta)_{23} + (\overline{\alpha\beta})_{2.})\underset{\sim}{1}_3 \\ (-(\alpha\beta)_{11} + (\overline{\alpha\beta})_{1.} + (\alpha\beta)_{21} - (\overline{\alpha\beta})_{2.})\underset{\sim}{1}_3 \\ (-(\alpha\beta)_{12} + (\overline{\alpha\beta})_{1.} + (\alpha\beta)_{22} - (\overline{\alpha\beta})_{2.})\underset{\sim}{1}_3 \\ (-(\alpha\beta)_{13} + (\overline{\alpha\beta})_{1.} + (\alpha\beta)_{23} - (\overline{\alpha\beta})_{2.})\underset{\sim}{1}_3 \end{bmatrix}$$

where

$$\bar{\alpha} = \frac{\alpha_1 + \alpha_2}{2}$$

$$\bar{\beta} = \frac{\beta_1 + \beta_2 + \beta_3}{3}$$

$$(\overline{\alpha\beta}) = \tfrac{1}{6} \sum_{i=1}^{2} \sum_{j=1}^{3} (\alpha\beta)_{ij}$$

$$(\overline{\alpha\beta})_{i.} = \frac{\alpha\beta_{i1} + \alpha\beta_{i2} + \alpha\beta_{i3}}{3}, \qquad i = 1,2$$

$$(\overline{\alpha\beta})_{.j} = \frac{\alpha\beta_{1j} + \alpha\beta_{2j}}{2}, \qquad j = 1,2,3.$$

Table 3.7 Some useful quantities used in the construction of the exact tests for the variance components utilizing the methodology developed in Section 3.3.3

Quantity	Formula cited	Value
ν	(3.189)	7
p	(3.189)	4
τ	(3.190)	(i,j,k)
c	(3.192)	18
n	(3.193)	40
b_4	Table 3.3	6
b_5	Table 3.3	3
b_6	Table 3.3	2
b_7	Table 3.3	1
P_4	Khuri(1990)	$\bar{J}_6 \otimes (I_3 - \bar{J}_3)$
P_5	Khuri(1990)	$(I_2 - \bar{J}_2) \otimes \bar{J}_3 \otimes (I_3 - \bar{J}_3)$
P_6	Khuri(1990)	$\bar{J}_2 \otimes (I_3 - \bar{J}_3) \otimes (I_3 - \bar{J}_3)$
P_7	Khuri(1990)	$(I_1 - \bar{J}_2) \otimes (I_3 - \bar{J}_3) \otimes (I_3 - \bar{J}_3)$
m_4	$m_4 = \text{rank}(P_4)$	2
m_5	$m_5 = \text{rank}(P_5)$	2
m_6	$m_6 = \text{rank}(P_6)$	4
m_7	$m_7 = \text{rank}(P_7)$	4
δ_4	(3.217)(or Table 3.3)	$6\sigma_4^2 + 3\sigma_5^2 + 2\sigma_6^2 + \sigma_7^2$
δ_5	(3.217)(or Table 3.3)	$3\sigma_5^2 + \sigma_7^2$
δ_6	(3.217)(or Table 3.3)	$2\sigma_6^2 + \sigma_7^2$

Table 3.7 – continued

Quantity	Formula cited	Value
δ_7	(3.217)(or Table 3.3)	σ_7^2
η_{r1}	(3.232)	12
η_{r2}	(3.232)	10
$\lambda_{\mathrm{r,max}}$	(3.237)	5/6
$\mathrm{SS}_{\mathrm{r4}} = \underset{\sim}{\omega}_4' \underset{\sim}{\omega}_4$	(3.124)	87.816
$\mathrm{SS}_{\mathrm{r5}} = \underset{\sim}{\omega}_5' \underset{\sim}{\omega}_5$	(3.124)	2.885
$\mathrm{SS}_{\mathrm{r6}} = \underset{\sim}{\omega}_7' \underset{\sim}{\omega}_7$	(3.124)	134.599
$\mathrm{SS}_{\mathrm{r7}} = \underset{\sim}{\omega}_7' \underset{\sim}{\omega}_7$	(3.124)	26.825
$\mathrm{SSE}_{\mathrm{r2}}$	(3.235)	31.5

Table 3.8 Expected mean square values for the cloth example

Source	Mean Square	Expected Value
C	$\mathrm{MS}_{\mathrm{r4}} = \mathrm{SS}_{\mathrm{r4}}/2$	$6\sigma_4^2 + 3\sigma_5^2 + 2\sigma_6^2 + \sigma_7^2 + \frac{5}{6}\sigma_\epsilon^2$
A*C	$\mathrm{MS}_{\mathrm{r5}} = \mathrm{SS}_{\mathrm{r5}}/2$	$3\sigma_5^2 + \sigma_7^2 + \frac{5}{6}\sigma_\epsilon^2$
B*C	$\mathrm{MS}_{\mathrm{r6}} = \mathrm{SS}_{\mathrm{r6}}/4$	$2\sigma_6^2 + \sigma_7^2 + \frac{5}{6}\sigma_\epsilon^2$
A*B*C	$\mathrm{MS}_{\mathrm{r7}} = \mathrm{SS}_{\mathrm{r7}}/4$	$\sigma_7^2 + \frac{5}{6}\sigma_\epsilon^2$
“Error”	$\mathrm{MSE}_{\mathrm{r2}} = \mathrm{SSE}_{\mathrm{r2}}/10$	$\frac{5}{6}\sigma_\epsilon^2$

Table 3.9 Results from the analysis of the variance components for the cloth example

Source	F-value	P-value
C	(exact test does not exist)	
A*C	$MS_{r5}/MS_{r7} = 0.215$	0.8152
B*C	$MS_{r6}/MS_{r7} = 5.018$	0.0737
A*B*C	$\frac{6}{5}(MS_{r7}/MSE_{r2}) = 2.555$	0.1044

Choosing $\underset{\sim}{L}' = (1, \underset{\sim}{0}'_{17})$, $\ell = 0$, and recalling Table 3.3, we see that to test $H_0: (\mu + \bar{\alpha} + \bar{\beta} + (\overline{\alpha\beta})) = 0$ we form $\left(\text{see } (3.286)\right)$

$$F_{H_0} = \frac{SSH_0/1}{MS_{f4}},$$

where, recalling (3.265) and (3.268),

$$MS_{f4} = \underset{\sim}{\omega}'_{f4}\underset{\sim}{\omega}_{f4}/2 = 49.204.$$

In this case, then, we find $F_{H_0} = 366.2$ with p-value $= 0.0027$ (based on 1 and 2 degrees of freedom).

To test $H_0: \alpha_1 + (\overline{\alpha\beta})_{1.} = \alpha_2 + (\overline{\alpha\beta})_{2.}$ we would take $\underset{\sim}{L}' = (1, \underset{\sim}{0}'_8, -1, \underset{\sim}{0}'_8)$ and $\ell = 0$. The appropriate test statistic would be

$$F_{H_1} = \frac{SSH_1/1}{MS_{f5}}$$

Table 3.10 Some key quantities used in the construction of the exact tests for the fixed-effects utilizing the methodology developed in Section 3.3.4

Quantity	Formula cited	Value
b_0	Table 3.3	18
b_1	Table 3.3	9
b_2	Table 3.3	6
b_3	Table 3.3	3
P_0	Khuri(1990)	$\bar{J}_{18}$
P_1	Khuri(1990)	$(I_2-\bar{J}_2)\otimes\bar{J}_9$
P_2	Khuri(1990)	$\bar{J}_2\otimes(I_3-\bar{J}_3)\otimes\bar{J}_3$
P_3	Khuri(1990)	$(I_2-\bar{J}_2)\otimes(I_3-\bar{J}_3)\otimes\bar{J}_3$
m_0	$m_0=\text{rank}(P_0)$	1
m_1	$m_1=\text{rank}(P_1)$	1
m_2	$m_2=\text{rank}(P_2)$	2
m_3	$m_3=\text{rank}(P_3)$	2
δ_0	(3.217)(or Table 3.3)	$6\sigma_4^2+3\sigma_5^2+2\sigma_6^2+\sigma_7^2$
δ_1	(3.217)(or Table 3.3)	$3\sigma_5^2+\sigma_7^2$
δ_2	(3.217)(or Table 3.3)	$2\sigma_6^2+\sigma_7^2$
δ_3	(3.217)(or Table 3.3)	σ_7^2
η_{f1}	(3.264)	18
$\lambda_{f,\max}$	(3.265)	1
SSH_0	(3.283)	18018.570
SSH_1	(3.283)	5.701
SSH_2	(3.283)	113.492
SSH_3	(3.283)	39.666

Table 3.11 Summary of results for the cloth example

Source	Test Statistic	p-value
mean	$SSH_0/MS_{f4} = 49.204$	0.0027
A	$SSH_1/MS_{f5} = 8.838$	0.0970
B	$SSH_2/2MS_{f6} = 1.479$	0.3305
A*C	$SSH_3/2MS_{f7} = 2020$	0.2476
C	$F_4^{\dagger} = 1.547$	0.3556
A*C	$MS_{r5}/MS_{r7} = 0.215$	0.8152
B*C	$MS_{r6}/MS_{r7} = 5.108$	0.0737
A*B*C	$\frac{6}{5}MS_{r7}/MSE_{r2} = 2.555$	0.1044

† Based on Satterthwaite's procedure.

where

$$MS_{f5} = \underset{\sim}{\omega}'_{f5}\underset{\sim}{\omega}_{f5}/2 = 0.645.$$

Thus $F_{H_1} = 8.838$ with a p-value of 0.0970. The degrees of freedom associated with this test were 1 and 2.

Similarly to test $H_0\colon \beta_1 + (\overline{\alpha\beta})_{.1} = \beta_2 + (\overline{\alpha\beta})_{.2} = \beta_3 + (\overline{\alpha\beta})_{.3}$ we take for our L matrix

$$L = \begin{bmatrix} (1, \underset{\sim}{0}'_5, -1, \underset{\sim}{0}'_{11}) \\ (1, \underset{\sim}{0}'_{11}, -1, \underset{\sim}{0}'_5) \end{bmatrix} \quad (\text{rank}(L) = 2).$$

Also $\underset{\sim}{\ell} = (0,0)'$. In the present case the value of the test statistic is

$$F_{H_2} = \frac{SSH_2/2}{MS_{f6}} = 1.479$$

with a significance level (based on 2 and 4 degrees of freedom) of 0.3305.

Finally, to test $H_0: (\overline{\alpha\beta})_{ij} - (\overline{\alpha\beta})_{i.} - (\overline{\alpha\beta})_{.j} + (\overline{\alpha\beta})_{..} = 0$ for all i and j, we form

$$F_{H_3} = \frac{SSH_3/2}{MS_{f7}} = 2.020$$

with a p-value of 0.2476 (based on 2 and 4 degrees of freedom). Table 3.11 summarizes our results.

Performance of machines. The data in Table 3.12 will be used to illustrate the methodology developed in Section 3.3.7. These data represent performance values for three machines. Two power settings (Factor A), three machines (Factor B), and three stations (Factor C) were selected and either two or three runs were made under each setting (see Montgomery (1984, pp. 377-378)). We assume that Factor A is fixed while Factors B and C are random. The appropriate model is given in (3.260).

The development of tests for the variance components follows similarly to that given in the previous example and thus we only summarize our results. This summary can be found in Table 3.13. We note that to test for a significant B or C effect we must use Satterthwaite's procedure (see Table 3.4).

Table 3.12 Performance values for three machines

Machine	1			2			3		
Station	1	2	3	1	2	3	1	2	3
	34.1	33.7	36.2	32.1	33.1	32.8	32.9	33.8	33.6
Power	30.3	34.9	37.1	33.5	33.9	35.1	33.1	32.8	32.8
Setting 1	31.6	35.0		34.0		34.3			31.7
Power	24.3	28.1	25.7	24.1	26.0	27.1	24.2	27.4	24.7
Setting 2	26.3	28.6	26.1	25.1	27.1	23.9	26.1	28.0	22.0
			24.9	27.9	23.9		25.3		24.8

(source: Montgomery (1984, pp. 377-378))

We next consider the fixed-effects. Recalling Table 3.4, Theorem 3.10, (3.318), and (3.321), we see that no exact test exists (based on our proposed procedures) for testing $H_0:\alpha_1 = \alpha_2$. In this example, the derived model is model (3.260) without the "$\epsilon_{ijk\ell}$" term. Thus, the fixed subscripts consist of just i, the nonnested random subscripts consist of j and k, and there are no nested random subscripts. Consequently, $s_1 = 1$, $s_2 = 2$, $s_3 = 0$, $s-1 = 3$. We also have $a_1 = 2$, $a_2 = a_3 = 3$. Furthermore,

$$d_1 = \prod_{i=s_1+1}^{s-1} a_i = a_2 a_3 = 9 \qquad \text{(see (3.329))}$$

Table 3.13 Results from the analysis of the variance components for the machine example

Source	df	MS	F-value	p-value
B	2	14.023	2.2209†	0.6889
A*B	2	11.801	0.7582	0.5256
C	2	18.629	0.7552†	0.6195
A*C	2	30.155	1.9375	0.2580
B*C	4	10.077	0.6475	0.6580
A*B*C	4	15.564	8.9481	0.0024
"Error"	10	1.739	——	——

†Based on Satterthwaite's Procedure

$$d_2 = 1 \qquad \left(\text{see } (3.330)\right)$$

$$\tau_0 = \min_{s_1+1 \le i \le s_1+s_2} a_i = 3 \qquad \left(\text{see } (3.371)\right)$$

$$\Lambda_0 = \{\underset{\sim}{\nu} : \underset{\sim}{\nu} = q\underset{\sim}{1}_2, \quad 1 \le q \le 3\} \qquad \left(\text{see } (3.370)\right)$$

$$H_0 = \underset{\sim}{1}_{18};\ H_1 = I_2 \otimes \underset{\sim}{1}_9 \qquad \left(\text{see } (3.334)\right).$$

The $\underset{\sim}{1}^{i_\ell}_{a_\ell}$ vectors $\left(\text{see } (3.338)\right)$ are restricted to $\ell = 2,3$ and to $i_\ell = q$ $(q = 1,2,3)$. Therefore, these vectors are (since $a_2 = a_3 = 3$)

$$\underset{\sim}{1}^1_3 = (1,0,0)',\ \underset{\sim}{1}^2_3 = (0,1,0)',\ \underset{\sim}{1}^3_3 = (0,0,1)'.$$

Table 3.14 Some useful quantities needed in the development of the exact confidence intervals for the machine example

Quantity	Formula cited	Value
$H_0^{\underset{\sim}{\nu}}, \ \underset{\sim}{\nu} = (1,1)'$	(3.338)	$\underset{\sim}{1}_2 \otimes \underset{\sim}{1}_3^1 \otimes \underset{\sim}{1}_3^1$
$H_0^{\underset{\sim}{\nu}}, \ \underset{\sim}{\nu} = (2,2)'$	(3.338)	$\underset{\sim}{1}_2 \otimes \underset{\sim}{1}_3^2 \otimes \underset{\sim}{1}_3^2$
$H_0^{\underset{\sim}{\nu}}, \ \underset{\sim}{\nu} = (3,3)'$	(3.338)	$\underset{\sim}{1}_2 \otimes \underset{\sim}{1}_3^3 \otimes \underset{\sim}{1}_3^3$
$H_1^{\underset{\sim}{\nu}}, \ \underset{\sim}{\nu} = (1,1)'$	(3.338)	$I_2 \otimes \underset{\sim}{1}_3^1 \otimes \underset{\sim}{1}_3^1$
$H_1^{\underset{\sim}{\nu}}, \ \underset{\sim}{\nu} = (2,2)'$	(3.338)	$I_2 \otimes \underset{\sim}{1}_3^1 \otimes \underset{\sim}{1}_3^2$
$H_1^{\underset{\sim}{\nu}}, \ \underset{\sim}{\nu} = (3,3)'$	(3.338)	$I_2 \otimes \underset{\sim}{1}_3^3 \otimes \underset{\sim}{1}_3^3$
$P_0^{\underset{\sim}{\nu}}, \ \underset{\sim}{\nu} \in \Lambda_0$	(3.347)	$H_0^{\underset{\sim}{\nu}} H_0^{\underset{\sim}{\nu}\prime}/18$
$P_1^{\underset{\sim}{\nu}}, \ \underset{\sim}{\nu} \in \Lambda_0$	(3.347)	$H_1^{\underset{\sim}{\nu}} H_1^{\underset{\sim}{\nu}\prime}/9 - H_0^{\underset{\sim}{\nu}} H_0^{\underset{\sim}{\nu}\prime}/18$
$P_{\omega 0}^{\underset{\sim}{\nu}}, \ \underset{\sim}{\nu} \in \Lambda_0$	(3.348)	$QP_0^{\underset{\sim}{\nu}}Q'$
$P_{\omega 1}^{\underset{\sim}{\nu}}, \ \underset{\sim}{\nu} \in \Lambda_0$	(3.348)	$QP_1^{\underset{\sim}{\nu}}Q'$
N_{00}	(3.351)	$\underset{\sim}{1}_2/9$
N_{01}	(3.351)	$J_2/18$
N_{10}	(3.351)	0
N_{11}	(3.351)	$I_2/9 - J_2/18$
N_0	(3.353)	$(2\underset{\sim}{1}_2 : J_2)/18$
N_1	(3.353)	$(0 : 2I_2 - J_2)/18$
C_0^*	(3.356)	$(1,0)$
C_1^*	(3.356)	$(1,0)$
B_0	(3.366)	$J_2/18$
B_1	(3.366)	$I_2/9 - J_2/18$
$d_2^2 C_0^* N_0 \underset{\sim}{g}$	Theorem 3.11	$(\mu + \bar{\alpha})/9$
$d_2^2 C_1^* N_1 \underset{\sim}{g}$	Theorem 3.11	$(\alpha_1 - \alpha_2)/18$

Table 3.14 - continued.

Quantity	Formula cited	Value
$d_2^2 C_0^* B_0 C_0^{*\prime}$	Theorem 3.11	1/18
$d_2^2 C_1^* B_1 C_1^{*\prime}$	Theorem 3.11	1/18
$\underset{\sim}{\bar{z}}_0$	(3.375)	3.5281
$\underset{\sim}{\bar{z}}_1$	(3.375)	0.7491
S_0	(3.374)	0.5185
S_1	(3.374)	0.6192

Some key quantities used in the development of the confidence intervals given in (3.379) are listed in Table 3.14. In this case, the space of estimable functions of the fixed-effects parameters is spanned by $C_0^* N_0 \underset{\sim}{g}$ and $C_1^* N_1 \underset{\sim}{g}$, or equivalently by

$$\mu+\bar{\alpha},\ \alpha_1-\alpha_2 \quad \text{(see Table 3.14)}$$

where $\bar{\alpha}=(\alpha_1+\alpha_2)/2$. In Table 3.15 we provide exact 95% confidence intervals for $\mu+\bar{\alpha}$ and $\alpha_1-\alpha_2$ along with approximate ones based on Satterthwaite's procedure. To apply Satterthwaite's procedure we note that (see Lemma 3.10 and Theorem 3.10)

$$\omega_0 \sim N\left(\sqrt{18}(\mu+\bar{\alpha}),\ \delta_0+\frac{\sigma_\epsilon^2}{2}\right)$$

Table 3.15 Exact and approximate 95% confidence intervals for the machine example

Estimable function	Exact confidence interval	Approximate confidence interval
$\mu+\bar{\alpha}$	(20.2696, 43.1367)	(24.9762, 36.3594)
$\alpha_1-\alpha_2$	(-11.3968, 38.3637)	(-31.8117, 12.9087)

$$\omega_1 \sim N\left(\frac{\sqrt{18}}{2}(\alpha_1-\alpha_2),\ \delta_1+\frac{\sigma_\epsilon^2}{2}\right)$$

where δ_0 and δ_1 are given in Table 3.4. Thus, an unbiased estimator of $\mu+\bar{\alpha}$ is $\omega_0/\sqrt{18}$ and an unbiased estimator of $\alpha_1-\alpha_2$ is $2\omega_1/\sqrt{18}$. From Table 3.4 we see that unbiased estimates of $(\delta_0+\frac{\sigma_\epsilon^2}{2})/18$ and $2(\delta_1+\frac{\sigma_\epsilon^2}{2})/9$ are, respectively,

$$\frac{1}{18}(MS_2+MS_4-MS_6) = 1.2541$$

$$\frac{2}{9}(MS_3+MS_5-MS_7) = 5.8648.$$

As a final remark we note that since there is a large A*B*C interaction, any conclusions which one reaches in regards to the other effects in the model must be done so cautiously.

3.3.9 Concluding Remarks

In this section we analyzed the general unbalanced mixed model with imbalance affecting the last stage only. We approached the problem of obtaining exact tests for the variance components and for estimable linear functions of the fixed-effects parameters in two different ways. In Subsections 3.3.3 and 3.3.4 we utilized resampling to first construct exact tests for the variance components and then to develop exact and approximate tests for the fixed-effects parameters. With the exception of the exact test for σ_{ν}^2, our exact tests for the variance components and the fixed-effects (when they exist) are Wald's tests. This is true since the degrees of freedom associated with these tests match those of the corresponding tests from the balanced data situation. A drawback to this approach was the need to use two different transformations to arrive at our exact procedures. An interesting side note to our methodology was the use of resampling as a precursor to utilizing Satterthwaite's procedure. Instead of (wrongly) applying Satterthwaite's methodology to sums of squares which were not independent nor distributed as scaled chi-squared variates, we first used our resampling scheme to transform the model into one in which the required assumptions were met, and then (correctly) applied Satterthwaite's procedure.

In Subsection 3.3.7 only one transformation was needed to arrive at a set of sums of squares which were independent and distributed as scaled chi-squared variates. These sums of squares mimicked their balanced counterparts. However, the resulting exact tests were seen to be less powerful (in general) than the exact tests developed in the two previously mentioned subsections. This led us to recommend the use of those tests when most of the cell counts are much larger than the minimum cell frequency.

Because the degrees of freedom associated with the exact tests for the variance components $\sigma^2_{\nu-p+1},\sigma^2_{\nu-p+2},\cdots,\sigma^2_{\nu-1}$, and for those fixed-effects parameters in which there existed a proper mean square that could serve as an "error" term in an F-ratio, match the degrees of freedom associated with the corresponding exact tests in a balanced mixed model, we can conclude that the exact tests developed in Subsection 3.3.7 are also Wald's tests. In cases where no error term existed, we utilized the results in Khuri (1984) and constructed exact simultaneous confidence intervals on linear functions of the fixed-effects parameters.

Finally, although not pursued in this dissertation, it would be very simple to construct simultaneous confidence intervals for continuous functions of the variance components by employing the methodology presented in Khuri (1981). This may be preferable in those situations where the only alternative is to use Satterthwaite's procedure (or some other approximate procedure) since the exact converage probabilities associated with these (approximate) intervals are typically unknown.

3.4 The Random Unbalanced Nested Model with Imbalance Affecting the Last Two Stages Only

3.4.1 Introduction

In this section we extend the results of Khuri (1987) by considering the random s-fold unbalanced nested model with imbalance affecting the last two stages only. We again utilize resampling to make transformations in a “step-down” manner to arrive at our exact tests. This approach is similar to the approach employed in Section 3.2. We assume all cells contain at least one observation.

After some preliminary development we proceed to the construction of the exact tests for the variance components in the model. Furthermore, we show that our tests reduce to the usual ANOVA tests when the design is balanced. Also, we develop expressions for the power associated with the tests. A numerical example is given to illustrate our proposed methodology.

3.4.2. Notations and Preliminary Development

In our development we will concentrate on the following model (the notation being similar to that of the previous section),

$$y_{\theta_{s+1}} = \mu + \gamma_{\theta_1} + \gamma_{\theta_2} + \cdots + \gamma_{\theta_s} + \epsilon_{\theta_{s+1}} \tag{3.380}$$

where the sets θ_j are defined as

$$\theta_j = (k_1, k_2, \cdots, k_j), \quad 1 \leq j \leq s+1 \tag{3.381}$$

and (since the design is balanced except for its last two stages)

$$k_j = \begin{cases} 1,2,\cdots,a_j, & j = 1,2,\cdots,s-1 \\ 1,2,\cdots,m_{\theta_{s-1}}, & j = s \\ 1,2,\cdots,n_{\theta_s}, & j = s+1. \end{cases} \tag{3.382}$$

In (3.381), θ_j represents the subscripts associated with the j^{th} effect in model (3.380), $j = 0, 1, \cdots, s+1$; (when $j = 0$, $\theta_j \equiv \phi$ and $\gamma_\phi \equiv \mu$). Furthermore we will assume that $\gamma_{\theta_1}, \gamma_{\theta_2}, \cdots, \gamma_{\theta_s}$, and $\epsilon_{\theta_{s+1}}$ are independently distributed with $\gamma_{\theta_j} \sim N(0, \sigma_j^2)$, $1 \leq j \leq s$; and $\epsilon_{\theta_{s+1}} \sim N(0, \sigma_\epsilon^2)$.

In matrix notation (3.380) becomes

$$\underset{\sim}{y} = \mu \underset{\sim}{1}_n + X_1 \underset{\sim}{\beta}_1 + \cdots + X_s \underset{\sim}{\beta}_s + \underset{\sim}{\epsilon}, \tag{3.383}$$

where $\underset{\sim}{y}$ is the vector of observations; $\underset{\sim}{\epsilon}$ is the vector of errors; $\underset{\sim}{\beta}_j$ is a random vector consisting of the elements of γ_{θ_j} $(1 \leq j \leq s)$; and X_j is a matrix of zeros and ones $(1 \leq j \leq s)$ written as

$$X_j = \bigoplus_{\theta_j} \underset{\sim}{1}_{n_{\theta_j}}, \quad 1 \leq j \leq s \tag{3.384}$$

where

$$n = \sum_{\theta_s} n_{\theta_s} \tag{3.385}$$

$$n_{\theta_j} = \sum_{\theta_s - \theta_j} n_{\theta_s}, \quad 1 \le j \le s-1, \tag{3.386}$$

where $\theta_s - \theta_j = (k_{j+1}, k_{j+2}, \cdots, k_s)$. Let us also define

$$m = \sum_{\theta_{s-1}} m_{\theta_{s-1}} \tag{3.387}$$

$$m_{\theta_j} = \sum_{\theta_{s-1} - \theta_j} m_{\theta_{s-1}}, \quad 1 \le j \le s-2, \tag{3.388}$$

where $\theta_{s-1} - \theta_j = (k_{j+1}, k_{j+2}, \cdots, k_{s-1})$. The matrix X_j is of order $n \times c_j$, $1 \le j \le s$, where

$$c_j = \begin{cases} \prod_{i=1}^{j} a_j, & j = 1,2,\cdots,s-1 \\ m, & j = s. \end{cases} \tag{3.389}$$

The assumptions made earlier concerning the random effects can now be restated as:

(i) $\underset{\sim}{\beta}_1, \underset{\sim}{\beta}_2, \cdots, \underset{\sim}{\beta}_s$, and $\underset{\sim}{\epsilon}$ are independent and normally distributed random vectors with zero mean vectors and variance-covariance matrices given by

$$\begin{aligned} \mathrm{Var}(\underset{\sim}{\beta}_i) &= \sigma_i^2 I_{c_i}, \quad i = 1,2,\cdots,s \\ \mathrm{Var}(\underset{\sim}{\epsilon}) &= \sigma_\epsilon^2 I_n. \end{aligned} \tag{3.390}$$

From (3.383) it is thus easy to verify that

$$E(\underset{\sim}{y}) = \mu \underset{\sim}{1}_n$$

$$\Omega = \operatorname{Var}(\underset{\sim}{y}) = \sum_{i=1}^{s} \sigma_i^2 X_i X_i' + \sigma_\epsilon^2 I_n . \qquad (3.391)$$

Besides the usual assumptions concerning the random effects we will also assume:

(i) The model used is model (3.380) (or model (3.383)).

(ii) $n > 2m - 1$. (3.392)

(iii) $m > 2c_{s-1} - 1$. (3.393)

The last two assumptions are needed for technical reasons to insure the validity of our methodology. Neither is overly restrictive. For example, (3.391) will hold if $n_{\theta_s} \geq 2$ for all $k_1, k_2, \cdots, k_s$ while (3.392) will hold if, for example, $m_{\theta_{s-1}} \geq 2$ for all $k_1, k_2, \cdots, k_{s-1}$.

3.4.3 An Exact Test for σ_s^2

Introduction. In this section we develop an exact test for the hypothesis $H_0 : \sigma_s^2 = 0$ vs $H_a : \sigma_s^2 \neq 0$. This test is similar to the corresponding test in a balanced model in that it compares the sum of squares associated with the γ_{θ_s} effect to the error sum of squares in model (3.380).

Preliminary development. To begin with let

$$W_j = [\underset{\sim}{1}_n : X_1 : \cdots : X_j], \quad 1 \leq j \leq s. \qquad (3.394)$$

Then, momentarily treating $\underset{\sim}{\beta}_1, \underset{\sim}{\beta}_2, \cdots, \underset{\sim}{\beta}_s$ as fixed, we define the sum of squares $R_{\underset{\sim}{y}}(\underset{\sim}{\beta}_s | \underset{\sim}{\beta}_1, \cdots, \underset{\sim}{\beta}_{s-1})$ as

$$R_{\underset{\sim}{y}}(\underset{\sim}{\beta}_s | \underset{\sim}{\beta}_1, \cdots, \underset{\sim}{\beta}_{s-1}) = \underset{\sim}{y}'[I_n - W_{s-1}(W'_{s-1}W_{s-1})^- W'_{s-1}]\underset{\sim}{y}$$

$$- \underset{\sim}{y}'[I_n - W_s(W'_sW_s)^- W'_s]\underset{\sim}{y}. \tag{3.395}$$

Since

$$\text{range}(\underset{\sim}{1}_n) \subset \text{range}(X_1) \subset \cdots \subset \text{range}(X_s), \tag{3.396}$$

we can express (3.395) as

$$R_{\underset{\sim}{y}}(\underset{\sim}{\beta}_s | \underset{\sim}{\beta}_1, \cdots, \underset{\sim}{\beta}_{s-1}) = \underset{\sim}{y}'[P_s - P_{s-1}]\underset{\sim}{y}, \tag{3.397}$$

where

$$P_j = X_j(X'_jX_j)^{-1}X'_j, \quad j = 1, 2, \cdots, s. \tag{3.398}$$

(From (3.384) it is clear that X_j has full column rank c_j, $j = 1, 2, \cdots, s-1$; and X_s has full column rank m.) $R_{\underset{\sim}{y}}(\underset{\sim}{\beta}_s | \underset{\sim}{\beta}_1, \cdots, \underset{\sim}{\beta}_{s-1})$ can be interpreted as the sum of squares for $\underset{\sim}{\beta}_s$ adjusted for $\underset{\sim}{\beta}_1, \cdots, \underset{\sim}{\beta}_{s-1}$ (see Searle (1971) for a description of the $R(\cdot|\cdot)$ notation).

The error sum of squares associated with model (3.380) (or model (3.383)) is given by

$$SSE_{\underset{\sim}{y}} = \underset{\sim}{y}'(I_n - W_s(W'_sW_s)^- W'_s)\underset{\sim}{y}$$

$$= \underset{\sim}{y}'(I_n - P_s)\underset{\sim}{y}. \qquad (3.399)$$

The last equality follows from (3.397). The following theorem provides the necessary tools to construct our exact test.

Theorem 3.12

(a) $R_{\underset{\sim}{y}}(\underset{\sim}{\beta}_s|\underset{\sim}{\beta}_1,\cdots,\underset{\sim}{\beta}_{s-1})$ is independent of $SSE_{\underset{\sim}{y}}$ and, under $H_0:\sigma_s^2 = 0$, is distributed as $\sigma_\epsilon^2\chi^2_{m-c_{s-1}}$.

(b) $SSE_{\underset{\sim}{y}}/\sigma_\epsilon^2 \sim \chi^2_{n-m}$.

Proof:

(a) We first verify that $R_{\underset{\sim}{y}}(\underset{\sim}{\beta}_s|\underset{\sim}{\beta}_1,\cdots,\underset{\sim}{\beta}_{s-1})$ is independent of $SSE_{\underset{\sim}{y}}$. To verify this we must show that

$$(P_s - P_{s-1})\Omega(I_n - P_s) = 0. \qquad (3.400)$$

In order to show (3.400) we need the following easily derived results:

(i) $X_iX_i'P_j = X_iX_i' = P_jX_iX_i', \quad 1 \le i \le j \le s$ (3.401)

(ii) $P_j^2 = P_j, \quad j = 1,2,\cdots,s.$ (3.402)

$$\text{(iii) } P_jP_i = P_j = P_iP_j, \quad 1 \le i \le j \le s. \tag{3.403}$$

For example to show (3.401) note that $\text{range}(X_iX_i') \subset \text{range}(P_j)$ for $i < j$ and equals $\text{range}(P_j)$ for $i = j$. Thus there exists a matrix X_{ij}^* such that

$$\begin{aligned} X_iX_i' &= P_jX_{ij}^* \\ &= X_{ij}^{*\prime}P_j \qquad \text{(by symmetry).} \end{aligned}$$

Consequently,

$$\begin{aligned} X_iX_i'P_j &= X_{ij}^{*\prime}P_j \qquad \Big(\text{by (3.402) which clearly holds}\Big) \\ &= X_iX_i' \\ &= P_jX_iX_i' \qquad \text{(by symmetry).} \end{aligned}$$

Formula (3.403) can be shown in a similar fashion.

Now, (3.401) implies $\Big($see (3.391)$\Big)$

$$\Omega(I_n - P_s) = (I_n - P_s)\sigma_\epsilon^2 \tag{3.404}$$

while (3.402) and (3.403) imply

$$(P_s - P_{s-1})(I_n - P_s) = 0. \tag{3.405}$$

Thus (3.400) is verified and consequently $SSE_{\underset{\sim}{y}}$ is independent of $R_{\underset{\sim}{y}}(\underset{\sim}{\beta}_s | \underset{\sim}{\beta}_1, \cdots, \underset{\sim}{\beta}_{s-1})$.

We now show that $R_{\underset{\sim}{y}}(\underset{\sim}{\beta}_s | \underset{\sim}{\beta}_1, \cdots, \underset{\sim}{\beta}_{s-1})/\sigma_\epsilon^2 \sim \chi^2_{m-c_{s-1}}$ when $\sigma_s^2 = 0$. To do so we need to verify that when $\sigma_s^2 = 0$ $(P_s - P_{s-1})\Omega/\sigma_\epsilon^2$ is idempotent; and $\operatorname{rank}(P_s - P_{s-1}) = m - c_{s-1}$. Now, from (3.391) and (3.401) we have

$$\begin{aligned}
(P_s - P_{s-1})\Omega &= (P_s - P_{s-1})\left[\sum_{i=1}^{s} \sigma_i^2 X_i X_i' + \sigma_\epsilon^2 I_n\right] \\
&= \left(\sum_{i=1}^{s} \sigma_i^2 X_i X_i' + \sigma_\epsilon^2 P_s\right) - \left(\sum_{i=1}^{s-1} \sigma_i^2 X_i X_i' \right. \\
&\qquad \left. + \sigma_s^2 P_{s-1} X_s X_s' + \sigma_\epsilon^2 P_{s-1}\right) \\
&= \sigma_s^2 X_s X_s' - \sigma_s^2 P_{s-1} X_s X_s' + \sigma_\epsilon^2 (P_s - P_{s-1}) \\
&= \sigma_s^2 P_s X_s X_s' - \sigma_s^2 P_{s-1} X_s X_s' + \sigma_\epsilon^2 (P_s - P_{s-1}) \ \Big(\text{see } (3.401)\Big) \\
&= (P_s - P_{s-1})\Big[\sigma_s^2 X_s X_s' + \sigma_\epsilon^2 I\Big].
\end{aligned} \tag{3.406}$$

Thus when $H_0 : \sigma_s^2 = 0$ holds, $(P_s - P_{s-1})\Omega/\sigma_\epsilon^2$ becomes

$$(P_s - P_{s-1})\Omega/\sigma_\epsilon^2 = P_s - P_{s-1}.$$

Since $P_s - P_{s-1}$ is idempotent $\Big($from (3.402) and (3.403)$\Big)$ and clearly has rank $m - c_{s-1}$, it follows that when $H_0:\sigma_s^2 = 0$ is true

$$R_{\underset{\sim}{y}}(\underset{\sim}{\beta}_s | \underset{\sim}{\beta}_1, \cdots, \underset{\sim}{\beta}_{s-1})/\sigma_\epsilon^2 \sim \chi^2_{m-c_{s-1}}.$$

(b) Recalling from (3.404) that

$$\Omega(I_n - P_s)/\sigma_\epsilon^2 = I_n - P_s$$

and using the fact that $I_n - P_s$ is idempotent and has rank $n - m$, it follows from (3.399) that

$$SSE_{\underset{\sim}{y}}/\sigma_\epsilon^2 \sim \chi^2_{n-m}. \qquad \square$$

The exact test. From Theorem 3.12, a test statistic for testing $H_0:\sigma_s^2 = 0$ vs $H_a:\sigma_s^2 \neq 0$ is given by

$$F_s = \frac{\underset{\sim}{y}'[P_s - P_{s-1}]\underset{\sim}{y}/(m-c_{s-1})}{SSE_{\underset{\sim}{y}}/(n-m)}. \tag{3.407}$$

Under H_0, $F_s \sim F_{m-c_{s-1},n-m}$. Under H_a the distribution of F_s is complicated. However, we still have

$$E\left[\frac{\underset{\sim}{y}'(P_s - P_{s-1})\underset{\sim}{y}}{m-c_{s-1}}\right] = \frac{1}{m-c_{s-1}}\left[\mathrm{tr}[(P_s - P_{s-1})\Omega] + \mu^2 \underset{=}{1}_n'(P_s - P_{s-1})\underset{=}{1}_n\right]. \tag{3.408}$$

Now, since $\mathrm{range}(\underset{=}{1}_n) \subset \mathrm{range}(P_j)$ and P_j is a projection matrix $\Big($recall (3.398)$\Big)$ we have that

$$P_j \underset{\sim}{1}_n = \underset{\sim}{1}_n, \quad j = 1,2,\cdots,s. \qquad (3.409)$$

Thus

$$\underset{\sim}{1}_n'(P_s - P_{s-1})\underset{\sim}{1}_n = 0.$$

Hence, from (3.406) and (3.409),

$$\mathrm{tr}[(P_s - P_{s-1})\Omega] = \sigma_\epsilon^2(m-c_{s-1}) + \sigma_s^2 \mathrm{tr}[X_s'(P_s - P_{s-1})X_s]$$

$$= \sigma_\epsilon^2(m - c_{s-1}) + \sigma_s^2 \sum_{j=1}^{m} \underset{\sim}{x}_{sj}'(P_s - P_{s-1})\underset{\sim}{x}_{sj}$$

where

$$X_s = [x_{s1}:x_{s2}:\cdots:x_{sm}].$$

Therefore,

$$E\left[\frac{\underset{\sim}{y}'(P_s - P_{s-1})\underset{\sim}{y}}{m-c_{s-1}}\right] = \sigma_\epsilon^2 + \frac{\sigma_s^2}{m-c_{s-1}} \sum_{j=1}^{m} \underset{\sim}{x}_{sj}'(P_s - P_{s-1})\underset{\sim}{x}_{sj} \geq \sigma_\epsilon^2. \qquad (3.410)$$

Consequently, we would reject H_0 for large values of F_s in (3.407). Note that resampling was not needed to construct this exact test.

We now develop an exact test for σ_{s-1}^2.

3.4.4 An Exact Test for σ^2_{s-1}

Introduction. In this section we develop an exact test for the hypothesis $H_0:\sigma^2_{s-1}=0$ vs. $H_a:\sigma^2_{s-1}\neq 0$. In the balanced data situation, we would simply compare, aside from a constant, $R_{\underset{\sim}{y}}(\underset{\sim}{\beta}_{s-1}|\underset{\sim}{\beta}_1,\cdots,\underset{\sim}{\beta}_{s-2})$ to $R_{\underset{\sim}{y}}(\underset{\sim}{\beta}_s|\underset{\sim}{\beta}_1,\cdots,\underset{\sim}{\beta}_{s-1})$ (treating $\underset{\sim}{\beta}_1,\cdots,\underset{\sim}{\beta}_s$ as fixed) where

$$R_{\underset{\sim}{y}}(\underset{\sim}{\beta}_{s-1}|\underset{\sim}{\beta}_1,\cdots,\underset{\sim}{\beta}_{s-2})=\underset{\sim}{y}'[P_{s-1}-P_{s-2}]\underset{\sim}{y},$$

(see (3.396) and (3.398)). Unfortunately, when the data are unbalanced, we can not, in general, show that

$$(P_{s-1}-P_{s-2})\Omega(P_s-P_{s-1})=0$$

unless we assume $\sigma^2_s=0$. This follows since

$$(P_{s-1}-P_{s-2})\Omega(P_s-P_{s-1})=\sigma^2_s(P_{s-1}-P_{s-2})X_sX_s'(P_s-P_{s-1}),$$

(recall (3.391) and use (3.401)-(3.403)). Thus these two sums of squares will not, in general, be independent unless we (unreasonably) assume $\sigma^2_s=0$. Consequently a different approach is necessary to test this hypothesis. We might try comparing the sum of squares for $\underset{\sim}{\beta}_{s-1}$ adjusted for all the other effects in the model (treating them as fixed) to $R_{\underset{\sim}{y}}(\underset{\sim}{\beta}_s|\underset{\sim}{\beta}_1,\cdots,\underset{\sim}{\beta}_{s-1})$. Unfortunately, the former sum of squares can be shown to zero (see (3.396)).

The idea we develop is based on a simple observation that was illustrated in Section 3.4.2:

> For any random nested model (balanced or unbalanced) we can construct an exact test for the variance component associated with the penultimate stage by comparing the mean square for the corresponding effect (adjusting for all the effects preceding it in the model) to the error mean square for the model.

Thus to test $H_0:\sigma^2_{s-1}=0$ vs. $H_a:\sigma^2_{s-1}\neq 0$ we will make a transformation (utilizing resampling) that will reduce model (3.380) $\big($or (3.383)$\big)$ to one that mimics a random (s-1)-fold nested model. In the transformed model, σ^2_{s-1} will be the variance component associated with the penultimate stage.

Preliminary development. Define the $c_s \times n$ matrix D as

$$
\begin{aligned}
D &= \underset{\theta_s}{\oplus}\frac{1}{n_{\theta_s}}\underset{\sim}{1}'_{n_{\theta_s}} \\
&= (X_s'X_s)^{-1}X_s'. \qquad (3.411)
\end{aligned}
$$

Also, let

$$\underset{\sim}{\bar{y}} = D\underset{\sim}{y}. \qquad (3.412)$$

Furthermore, define

$$A_j = \bigoplus_{\theta_j} \underset{\sim}{1}_{m_{\theta_j}}, \qquad j = 1,2,\cdots,s-1. \tag{3.413}$$

The matrix A_j is or order $m \times c_j$ and of full column rank c_j $(j = 1,2,\cdots,s-1)$. From (3.384) and (3.396) we can show that

$$\begin{aligned} \underset{\sim}{1}_n &= X_s \underset{\sim}{1}_m \\ X_j &= X_s A_j, \qquad j = 1,2,\cdots,s-1. \end{aligned} \tag{3.414}$$

Consequently, (3.412) can be expressed as

$$\bar{\underset{\sim}{y}} = \mu \underset{\sim}{1}_m + A_1 \underset{\sim}{\beta}_1 + A_2 \underset{\sim}{\beta}_2 + \cdots + A_{s-1} \underset{\sim}{\beta}_{s-1} + I_m \underset{\sim}{\beta}_s + \bar{\underset{\sim}{\epsilon}} \tag{3.415}$$

where $\bar{\underset{\sim}{\epsilon}} = D \underset{\sim}{\epsilon}$. We note that

$$\text{range}(\underset{\sim}{1}_m) \subset \text{range}(A_1) \subset \cdots \subset \text{range}(A_{s-1}). \tag{3.416}$$

This follows since (writing $X_0 = \underset{\sim}{1}_n$ and $A_0 = \underset{\sim}{1}_m$), for $0 \le i < j \le s-1$,

$$\begin{aligned} \text{range}(A_i) &= \text{range}(X_s A_i) && (X_s \text{ has full column rank}) \\ &= \text{range}(X_i) && \big(\text{see } (3.414)\big) \\ &\subset \text{range}(X_j) && \big(\text{see } (3.396)\big) \end{aligned}$$

$$= \text{range}(X_s A_j)$$

$$= \text{range}(A_j).$$

From (3.415) it is easily seen that

$$E(\underset{\sim}{y}) = \mu \underset{\sim}{1}_m$$

$$\Sigma = \text{Var}(\underset{\sim}{y}) = \sum_{i=1}^{s-1} \sigma_i^2 A_i A_i' + \sigma_s^2 I_m + K\sigma_\epsilon^2, \tag{3.417}$$

where

$$K = (X_s' X_s)^{-1} = \underset{\theta_s}{\oplus} n_{\theta_s}^{-1}. \tag{3.418}$$

Now, let H_1 consist of the last m-1 rows of the m×m Helmert matrix (see Searle (1982, p. 71)). Then H_1 has the following properties:

(i) $H_1 H_1' = I_{m-1}$.

(ii) $H_1' H_1 = I_m - \bar{J}_m$. (3.419)

(iii) $H_1 \underset{\sim}{1}_m = \underset{\sim}{0}_{m-1}$.

(Recall that $\bar{J}_m = \frac{1}{m} \underset{\sim}{1}_m \underset{\sim}{1}_m'$.) Define the $(m-1) \times 1$ random vector $\underset{\sim}{\mu}$ as

$$\underset{\sim}{u} = H_1 \underset{\sim}{y} \tag{3.420}$$

The purpose of this transformation is to cause $\underset{\sim}{u}$ to have a zero mean vector as illustrated below (see (3.417)).

$$E(\underset{\sim}{u}) = H_1 E(\underset{\sim}{\bar{y}}) = \mu H_1 \underset{\sim}{1}_m = \underset{\sim}{0}_{m-1}$$

$$\mathrm{Var}(\underset{\sim}{u}) = \sum_{i=1}^{s-1} H_1 \tilde{A}_i H_1' \sigma_i^2 + \sigma_s^2 I_{m-1} + H_1 K H_1' \sigma_\epsilon^2, \tag{3.421}$$

where $\tilde{A}_i = A_i A_i'$, $i = 1, 2, \cdots, s-1$.

Utilizing resampling. Let R be the $n \times n$ matrix associated with error sum of squares in model (3.380). That is, from (3.399),

$$R = I_n - P_s. \tag{3.422}$$

The matrix R is symmetric, idempotent, and has rank n-m (see Lemma B.4). We can write R as

$$R = C \Lambda C' \tag{3.423}$$

where C is orthogonal and Λ is diagonal. The first n-m diagonal entries of Λ are ones while the remaining m diagonal entries are zeros. Partition C and Λ as

$$C = [C_1 : C_2 : C_3]$$

$$\Lambda = \mathrm{diag}(I_{\eta_1}, I_{\eta_2}, 0) \tag{3.424}$$

where

$$\eta_1 = m-1$$

$$\eta_2 = n-2m+1 \qquad (3.425)$$

and C_1, C_2, and C_3 are of orders $n \times \eta_1$, $n \times \eta_2$, and $n \times m$, respectively. Note that $\eta_1 + \eta_2 = n-m = \text{rank}(R)$ and $\eta_2 > 0$ by (3.392).

We now introduce a transformation that will result in a random vector which has a diagonal variance-covariance matrix. Let $\underset{\sim}{\omega}$ be the $(m-1) \times 1$ random vector defined as

$$\underset{\sim}{\omega} = \underset{\sim}{u} + (\lambda_{\max} I_{\eta_1} - L)^{\frac{1}{2}} C_1' \underset{\sim}{y} \qquad (4.426)$$

where

$$L = H_1 K H_1' \qquad (4.427)$$

and $\lambda_{\max}$ is its largest eigenvalue. The matrix $(\lambda_{\max} I_{\eta_1} - L)^{\frac{1}{2}}$ is well-defined since $\lambda_{\max} I_{\eta_1} - L$ is p.s.d. It's eigenvalues are the square roots of the eigenvalues of $\lambda_{\max} I_{\eta_1} - L$. The next lemma provides some distributional properties for $\underset{\sim}{\omega}$.

Lemma 3.13

The random vector $\underset{\sim}{\omega}$ is normally distributed with

(a) $E(\underset{\sim}{\omega}) = \underset{\sim}{0}$

(3.428)

(b) $\text{Var}(\underset{\sim}{\omega}) = \sum_{i=1}^{s-1} H_1 \tilde{A}_i H_1' \sigma_i^2 + \delta I_{m-1}$, where $\delta = \sigma_s^2 + \lambda_{\max} \sigma_\epsilon^2$.

Proof:

First note that $\underset{\sim}{\omega}$ is clearly normally distributed.

(a) From (3.421), $E(\underset{\sim}{u}) = \underset{\sim}{u}$. From (3.398), (3.414), and (3.422), it follows that $R\underset{\sim}{1}_n = \underset{\sim}{0}$. Thus, since $\text{range}(C_1) \subset \text{range}(R)$, it must be that $C_1'\underset{\sim}{1}_n = \underset{\sim}{0}$. Consequently the result follows after recalling the definition of $\underset{\sim}{\omega}$ in (4.426) and that $E(\underset{\sim}{y}) = \mu\underset{\sim}{1}_n$.

(b) We claim that $\underset{\sim}{u}$ is independent of $C_1'\underset{\sim}{y}$ in (4.426). To see this first note that

$$\begin{aligned} D\Omega R &= DR\sigma_\epsilon^2 && \Big(\text{see } (3.404)\Big) \\ &= 0. && \Big(\text{see } (3.398),\ (3.411),\ \text{and } (3.422)\Big) \end{aligned}$$

Thus, since $\text{range}(C_1) \subset \text{range}(R)$ it follows that $D\Omega C_1 = 0 \Rightarrow \underset{\sim}{\bar{y}}$ is independent of $C_1'\underset{\sim}{y}$. But this implies that $\underset{\sim}{u}$ and $C_1'\underset{\sim}{y}$ are independent since $\underset{\sim}{u}$ is a function of $\underset{\sim}{\bar{y}}$. Consequently,

$$\text{Var}(\underset{\sim}{\omega}) = \text{Var}(\underset{\sim}{u}) + (\lambda_{\max} I_{\eta_1} - L)^{\frac{1}{2}} C_1'\Omega C_1 (\lambda_{\max} I_{\eta_1} - L)^{\frac{1}{2}}.$$

Now, from (3.391), (3.401), (3.422), and the fact that R is idempotent, we have

$$R\Omega R = R\sigma_\epsilon^2.$$

Thus, since $\text{range}(C_1) \subset \text{range}(R)$, it follows that

$$C_1'\Omega C_1 = \sigma_\epsilon^2 C_1'C_1 = \sigma_\epsilon^2 I_{\eta_1}.$$

Hence, from (3.421),

$$\begin{aligned}\text{Var}(\underset{\sim}{\omega}) &= \sum_{i=1}^{s-1} H_1\tilde{A}_i H_1'\sigma_i^2 + \sigma_s^2 I_{m-1} + L\sigma_\epsilon^2 + (\lambda_{\max} I_{\eta_1} - L)\sigma_\epsilon^2 \\ &= \sum_{i=1}^{s-1} H_1\tilde{A}_i H_1'\sigma_i^2 + \delta I_{m-1}\end{aligned}$$

where $\delta = \sigma_s^2 + \lambda_{\max}\sigma_\epsilon^2$. □

From this lemma, (3.415), and (3.420), we can write $\underset{\sim}{\omega}$ as

$$\underset{\sim}{\omega} = H_1A_1\underset{\sim}{\beta}_1 + H_1A_2\underset{\sim}{\beta}_2 + \cdots + H_1A_{s-1}\underset{\sim}{\beta}_{s-1} + \epsilon^* \tag{3.429}$$

where $\underset{\sim}{\epsilon}^* = H_1\underset{\sim}{\beta}_s + H_1\underset{\sim}{\bar{\epsilon}} + (\lambda_{\max} I_{\eta_1} - L)^{\frac{1}{2}} C_1'\underset{\sim}{\epsilon} \sim N(\underset{\sim}{0}, \delta I_{m-1})$ independently of $\underset{\sim}{\beta}_1, \cdots, \underset{\sim}{\beta}_{s-1}$. Model (3.429) resembles a random (s-1)-fold nested model (with zero intercept) because of the distributional properties of $\underset{\sim}{\beta}_1, \cdots, \underset{\sim}{\beta}_{s-1}$, and $\underset{\sim}{\epsilon}^*$, and since for $1 \le i < j \le s-1$,

$$H_1A_i = H_1A_jA_{ij}^* \quad \Big(\text{for some matrix } A_{ij}^*, \text{ see } (3.416)\Big)$$

$$\Rightarrow \text{range}(H_1A_i) \subset \text{range}(H_1A_j). \tag{3.430}$$

Before we proceed to the development of our exact test we need the following result.

Lemma 3.14

(a) $\text{rank}(A_j'H_1'H_1A_j) = c_j - 1, \quad j = 1,2,\cdots,s\text{-}1.$

(b) A g-inverse of $A_j'H_1'H_1A_j$ is given by

$$(A_j'H_1'H_1A_j)^- = (A_j'A_j)^{-1}, \quad j = 1,2,\cdots,s\text{-}1.$$

Proof:

(a) $$\text{rank}(A_j'H_1'H_1A_j) = \text{rank}(H_1A_j)$$
$$\leq \text{rank}(A_j) = c_j.$$

However, from (3.413) $A_j\underset{\sim}{1}_{c_j} = \underset{\sim}{1}_m$. Since $H_1\underset{\sim}{1}_m = \underset{\sim}{0}_{m-1}$ $\Big($see (3.419)(iii)$\Big)$, the columns of H_1A_j are linearly dependent. Thus $\text{rank}(A_j'H_1'H_1A_j) \leq c_j - 1, \quad j = 1,2,\cdots,s\text{-}1.$ Conversely, by the Frobenius Inequality (see Lemma A.1), we have

$$\text{rank}(H_1A_j) \geq \text{rank}(H_1) + \text{rank}(A_j) - \text{rank}(I_m)$$

$$= m - 1 + c_j - m$$

$$= c_j - 1, \quad j = 1,2,\cdots,s\text{-}1.$$

Thus, $\mathrm{rank}(A'_j H'_1 H_1 A_j) = c_j - 1, \quad j = 1,2,\cdots,s\text{-}1,$

(b) First note that $\Big($see (3.413)$\Big)$ for $j = 1,2,\cdots,s\text{-}1,$

$$(A_j(A'_jA_j)^{-1}A'_j)\bar{J}_m = \Big(\underset{\theta_j}{\oplus}\frac{1}{m_{\theta_j}}J_{m_{\theta_j}}\Big)\bar{J}_m$$

$$= \bar{J}_m \tag{3.431}$$

$$= \bar{J}_m(A_j(A'_jA_j)^{-1}A'_j). \tag{3.432}$$

Hence, for $j = 1,2,\cdots,s\text{-}1$

$$A'_jH'_1H_1A_j(A'_jA_j)^{-1}A'_jH'_1H_1A_j$$

$$= A'_j(I_m - \bar{J}_m)A_j(A'_jA_j)^{-1}A'_j(I_m - \bar{J}_m)A_j \quad \Big(\text{see } (3.419)\Big)$$

$$= (A'_j - A'_j\bar{J}_mA_j(A'_jA_j)^{-1}(I_m - \bar{J}_m)A_j$$

$$= (A'_j - A'_j\bar{J}_m)(I_m - \bar{J}_m)A_j \quad \Big(\text{see } (3.431) \text{ and } (3.432)\Big)$$

$$= A'_j(I_m - \bar{J}_m)A_j \qquad (I_m - \bar{J}_m \text{ is idempotent})$$

$$= A'_jH'_1H_1A_j. \qquad \square$$

We now develop the exact test for σ^2_{s-1}.

The exact test. Let us define the $(m-1)\times(m-1)$ matrix Q_j $(j=1,2,\cdots,s-1)$ as

$$Q_j = H_1A_j(A'_jH'_1H_1A_j)^-H'_1$$

$$= H_1A_j(A'_jA_j)^{-1}A'_jH'_1. \qquad (4.433)$$

Furthermore let $R_{\underset{\sim}{\omega}}(\underset{\sim}{\beta}_{s-1}|\underset{\sim}{\beta}_1,\cdots,\underset{\sim}{\beta}_{s-2})$ be the sum of squares for the $\underset{\sim}{\beta}_{s-1}$ effect adjusting for $\underset{\sim}{\beta}_1,\cdots,\underset{\sim}{\beta}_{s-2}$ (treating them as fixed) for the $\underset{\sim}{\omega}$ model defined in (3.429). Also let $SSE_{\underset{\sim}{\omega}}$ be the error sum of squares for model (3.429). Then

$$R_{\underset{\sim}{\omega}}(\underset{\sim}{\beta}_{s-1}|\underset{\sim}{\beta}_1,\cdots,\underset{\sim}{\beta}_{s-2}) = \underset{\sim}{\omega}'[I_{m-1}-V_{s-2}(V'_{s-2}V_{s-2})^-V'_{s-2}]\underset{\sim}{\omega}$$

$$-\underset{\sim}{\omega}'[I_{m-1}-V_{s-1}(V'_{s-1}V_{s-1})^-V'_{s-1}]\underset{\sim}{\omega}$$

and

$$SSE_{\underset{\sim}{\omega}} = \underset{\sim}{\omega}'(I_{m-1}-V_{s-1}(V'_{s-1}V_{s-1})^-V'_{s-1})\underset{\sim}{\omega}$$

where

$$V_j = [H_1A_1:H_1A_2:\cdots:H_1A_j],\ j=1,2,\cdots,s-1. \qquad (3.434)$$

From (3.430), Lemma 3.14(b), and (3.433), $R_{\underset{\sim}{\omega}}(\underset{\sim}{\beta}_{s-1}|\underset{\sim}{\beta}_1,\cdots,\underset{\sim}{\beta}_{s-2})$ and $SSE_{\underset{\sim}{\omega}}$ can be equivalently expressed as

$$R_{\underset{\sim}{\omega}}(\underset{\sim}{\beta}_{s-1}|\underset{\sim}{\beta}_1,\cdots,\underset{\sim}{\beta}_{s-2}) = \underset{\sim}{\omega}'\ [Q_{s-1} - Q_{s-2}]\underset{\sim}{\omega} \tag{3.435}$$

$$SSE_{\underset{\sim}{\omega}} = \underset{\sim}{\omega}'[I_{m-1} - Q_{s-1}]\underset{\sim}{\omega}. \tag{3.436}$$

The major result of this section is given in the next Theorem which provides some distributional properties for $R_{\underset{\sim}{\omega}}(\underset{\sim}{\beta}_{s-1}|\underset{\sim}{\beta}_1,\cdots,\underset{\sim}{\beta}_{s-2})$ and $SSE_{\underset{\sim}{\omega}}$.

Theorem 3.13

(a) $R_{\underset{\sim}{\omega}}(\underset{\sim}{\beta}_{s-1}|\underset{\sim}{\beta}_1,\cdots,\underset{\sim}{\beta}_{s-2})$ is independent of $SSE_{\underset{\sim}{\omega}}$ and, under $H_0{:}\sigma^2_{s-1} = 0$, is distributed as $\delta\chi^2_{c_{s-1} - c_{s-2}}$.

(b) $SSE_{\underset{\sim}{\omega}}/\delta \sim \chi^2_{m-c_{s-1}}$, $\left(\delta \text{ was given in Lemma 3.13(b)}\right)$.

Proof:

(a) We first verify that $R_{\underset{\sim}{\omega}}(\underset{\sim}{\beta}_{s-1}|\underset{\sim}{\beta}_1,\cdots,\underset{\sim}{\beta}_{s-2})$ and $SSE_{\underset{\sim}{\omega}}$ are independent by showing

$$[Q_{s-1} - Q_{s-2}][\mathrm{Var}(\underset{\sim}{\omega})][I_{m-1} - Q_{s-1}] = 0.$$

In order to show this we need the following results, which are analogous to (3.401) - (3.403).

(i) $$H_1A_iA_i'H_1'Q_j = H_1A_iA_i'H_1' = Q_jH_1A_iA_i'H_1', \quad 1 \le i \le j \le s-1 \tag{3.437}$$

(ii) $Q_j^2 = Q_j, \quad j = 1,2,\cdots,s-1$ (3.438)

(iii) $Q_j Q_i = Q_j = Q_i Q_j, \quad 1 \leq i \leq j \leq s-1$ (3.439)

Formula (3.438) is obviously true. To show (3.437) note that for $1 \leq i \leq j \leq s-1$

$$\begin{aligned} \text{range}(H_1 A_i A_i' H_1') &= \text{range}(H_1 A_i) \\ &\subset \text{range}(H_1 A_j) \qquad \big(\text{see } (3.430)\big) \\ &= \text{range}(Q_j). \qquad \big(\text{see } (3.433)\big) \end{aligned}$$

Thus there exists a matrix B_{ij}^* such that

$$\begin{aligned} H_1 A_i A_i' H_1' &= Q_j B_{ij}^* \\ &= B_{ij}^{*\prime} Q_j. \qquad (\text{by symmetry}) \end{aligned}$$

Hence, for $1 \leq i \leq j \leq s-1$,

$$\begin{aligned} H_1 A_i A_i' H_1' Q_j &= B_{ij}^{*\prime} Q_j^2 \\ &= B_{ij}^{*\prime} Q_j \qquad \big(\text{by } (3.438)\big) \\ &= H_1 A_i A_i' H_1' \end{aligned}$$

$$= Q_j H_1 A_i A_i' H_1' \qquad \text{(by symmetry)}.$$

Formula (3.439) follows similarly. Now, from Lemma 3.13(b), we have

$$\begin{aligned}
\mathrm{Var}(\underset{\sim}{\omega})[I_{m-1} - Q_{s-1}] &= \Big(\sum_{i=1}^{s-1} H_1 \tilde{A}_i H_1' \sigma_i^2 + \delta I_{m-1}\Big)(I_{m-1} - Q_{s-1}) \\
&= \sum_{i=1}^{s-1} H_1 \tilde{A}_i H_1' \sigma_i^2 + \delta I_{m-1} \\
&\quad - \Big(\sum_{i=1}^{s-1} H_1 \tilde{A}_i H_1' Q_{s-1} + \delta Q_{s-1}\Big) \\
&= \delta(I_{m-1} - Q_{s-1}) \qquad \Big(\text{see } (3.437)\Big). \qquad (3.440)
\end{aligned}$$

Thus

$$\begin{aligned}
&[Q_{s-1} - Q_{s-2}][\mathrm{Var}(\underset{\sim}{\omega})][I_{m-1} - Q_{s-1}] \\
&= \delta(Q_{s-1} - Q_{s-2})(I_{m-1} - Q_{s-1}) \\
&= 0 \qquad \Big(\text{by } (3.349)\Big).
\end{aligned}$$

Consequently, $R_{\underset{\sim}{\omega}}(\underset{\sim}{\beta}_{s-1} | \underset{\sim}{\beta}_1, \cdots, \underset{\sim}{\beta}_{s-2})$ and $SSE_{\underset{\sim}{\omega}}$ are independent.

To show that $R_{\underset{\sim}{\omega}}(\underset{\sim}{\beta}_{s-1} | \underset{\sim}{\beta}_1, \cdots, \underset{\sim}{\beta}_{s-2})/\delta$ has a chi-squared distribution with $c_{s-1} - c_{s-2}$ degrees of freedom under $H_0: \sigma_{s-1}^2 = 0$, we need to verify that $(Q_{s-1} - Q_{s-2})(\mathrm{Var}(\underset{\sim}{\omega}))/\delta$ is idempotent (when $\sigma_{s-1}^2 = 0$) and $Q_{s-1} - Q_{s-2}$ has rank $c_{s-1} - c_{s-2}$. Now from Lemma 3.13(b) and (3.437), we have

$$(Q_{s-1} - Q_{s-2}) \left[\sum_{i=1}^{s-1} H_1 \tilde{A}_i H_1' \sigma_i^2 + I_{m-1} \delta \right]$$

$$= \sum_{i=1}^{s-1} H_1 \tilde{A}_i H_1' \sigma_i^2 + \delta Q_{s-1} - \left(\sum_{i=1}^{s-2} H_1 \tilde{A}_i H_1' \sigma_i^2 + \sigma_{s-1}^2 Q_{s-2} H_1 \tilde{A}_{s-1} H_1' + \delta Q_{s-2} \right)$$

$$= \sigma_{s-1}^2 Q_{s-1} H_1 \tilde{A}_{s-1} H_1' - \sigma_{s-1}^2 Q_{s-2} H_1 \tilde{A}_{s-1} H_1' + \delta (Q_{s-1} - Q_{s-2}) \Big(\text{see } (3.437)\Big)$$

$$= (Q_{s-1} - Q_{s-2}) [\sigma_{s-1}^2 H_1 \tilde{A}_{s-1} H_1' + \delta I_{m-1}].$$

Thus when $H_0 : \sigma_{s-1}^2 = 0$ holds, $(Q_{s-1} - Q_{s-2})(\mathrm{Var}(\underset{\sim}{\omega}))/\delta$ becomes

$$(Q_{s-1} - Q_{s-2})(\mathrm{Var}(\underset{\sim}{\omega}))/\delta = Q_{s-1} - Q_{s-2}$$

Since $Q_{s-1} - Q_{s-2}$ is idempotent (from (3.348) and (3.349)) and clearly has rank $c_{s-1} - c_{s-2}$ (see (3.433) and Lemma 3.14(a)), it follows that when $H_0 : \sigma_{s-1}^2 = 0$ is true

$$R_{\underset{\sim}{\omega}}(\underset{\sim}{\beta}_{s-1} | \underset{\sim}{\beta}_1, \cdots, \underset{\sim}{\beta}_{s-2})/\delta \sim \chi^2_{c_{s-1} - c_{s-2}}.$$

(b) From (3.440),

$$\mathrm{Var}(\underset{\sim}{\omega})[I_{m-1} - Q_{s-1}] = \delta (I_{m-1} - Q_{s-1})$$

Thus $\mathrm{Var}(\underset{\sim}{\omega})[I_{m-1} - Q_{s-1}]/\delta$ is idempotent. Since $\mathrm{rank}(I_{m-1} - Q_{s-1}) = m - 1 - (c_{s-1} - 1) = m - c_{s-1}$ (recall (4.333) and Lemma 3.14(a)), it follows that (see (3.436))

$$\mathrm{SSE}_{\omega}/\delta \sim \chi^2_{m - c_{s-1}}. \qquad \square$$

To test $H_0:\sigma^2_{s-1}=0$ vs. $H_a:\sigma^2_{s-1}\neq 0$ we form

$$F_{s-1}=\frac{\underset{\sim}{\omega}'(Q_{s-1}-Q_{s-2})\underset{\sim}{\omega}/(c_{s-1}-c_{s-2})}{SSE_{\underset{\sim}{\omega}}/(m-c_{s-1})}. \tag{3.441}$$

Under H_0, $F_{s-1}\sim F_{c_{s-1}-c_{s-2},m-c_{s-1}}$. Under H_a, the distribution is complicated. However, we still have

$$E\left[\frac{\underset{\sim}{\omega}'(Q_{s-1}-Q_{s-2})\underset{\sim}{\omega}}{c_{s-1}-c_{s-2}}\right]=\frac{1}{c_{s-1}-c_{s-2}}\ \mathrm{tr}\left[(Q_{s-1}-Q_{s-2})(\mathrm{Var}(\underset{\sim}{\omega}))\right]$$

$$=\frac{1}{c_{s-1}-c_{s-2}}\mathrm{tr}\left[(Q_{s-1}-Q_{s-2})\left(\sum_{i=1}^{s-1}H_1\tilde{A}_iH_1'\sigma_i^2+\delta I_{m-1}\right)\right]\quad \left(\text{see Lemma 3.13(b)}\right)$$

$$=\frac{1}{c_{s-1}-c_{s-2}}\mathrm{tr}[\sigma^2_{s-1}Q_{s-1}H_1\tilde{A}_{s-1}H_1'-\sigma^2_{s-1}Q_{s-2}H_1\tilde{A}_{s-1}H_1'$$

$$+\delta(Q_{s-1}-Q_{s-2})]\quad \left(\text{see } (3.347)\right)$$

$$=\delta+\frac{\sigma^2_{s-1}}{c_{s-1}-c_{s-2}}\mathrm{tr}\left[(Q_{s-1}-Q_{s-2})H_1\tilde{A}_{s-1}H_1'\right]$$

$$=\delta+\frac{\sigma^2_{s-1}}{c_{s-1}-c_{s-2}}\ \mathrm{tr}\left[A'_{s-1}H_1'(Q_{s-1}-Q_{s-2})H_1A_{s-1}\right]$$

$$=\delta+\frac{\sigma^2_{s-1}}{c_{s-1}-c_{s-2}}\sum_{j=1}^{c_{s-1}}a^{*\prime}_{s-1,j}(Q_{s-1}-Q_{s-2})a^*_{s-1,j}$$

where $a^*_{s-1,j}=H_1\underset{\sim}{a}_{s-1,j}$, $j=1,2,\cdots,c_{s-1}$, and

$$A_{s-1}=\left[\underset{\sim}{a}_{s-1,1}:\underset{\sim}{a}_{s-1,2}:\cdots:\underset{\sim}{a}_{s-1,c_{s-1}}\right].$$

Thus we reject H_0 for large values of F_{s-1} in (3.441).

3.4.5. Exact Tests Concerning $\sigma_1^2, \sigma_2^2, \cdots, \sigma_{s-2}^2$

Introduction. Recall that model (3.429) mimics a random unbalanced (s-1)-fold nested model where the imbalance affects the last stage only. Thus we can adopt the procedures developed in Khuri (1990) to aid us in the construction of our exact tests. In that article, Khuri was able to develop his exact procedures by first averaging over the last subscript and then making a transformation that resulted in a random vector having a zero mean vector. However this new vector had a variance-covariance structure that was not diagonal (see Khuri (1990, p. 182)). To obtain a diagonal variance-covariance matrix, Khuri made another transformation, this time based on a resampling scheme. Thus these transformations effectively reduced the analysis of the unbalanced model to that of a balanced one.

We can employ the same ideas to our present situation. However, some care needs to be taken since model (3.429) is not exactly (in a statistical modeling sense) a random unbalanced nested model. This is true since the H_1A_j $(j=0,1,\cdots,s-1)$ are not design matrices.

Preliminary development. Recall model (3.429). Namely,

$$\underset{\sim}{\omega} = H_1A_1\underset{\sim}{\beta}_1 + \cdots + H_1A_{s-1}\underset{\sim}{\beta}_{s-1} + \underset{\sim}{\epsilon}^*.$$

Define the $c_{s-1} \times (m-1)$ matrix D^* as

$$D^* = (A'_{s-1}H'_1H_1A_{s-1})^- A'_{s-1}H'_1$$

$$= (A'_{s-1}A_{s-1})^{-1}A'_{s-1}H'_1 \qquad \Big(\text{see Lemma 3.14(b)}\Big). \qquad (3.442)$$

Compare the definition of D^* to that of D given in (3.411). The matrix D was used in (3.412) to average over the last subscript in model (3.380) $\Big($or model (3.383)$\Big)$. Similarly to (3.412), define the $c_{s-1} \times 1$ random vector $\bar{\underset{\sim}{\omega}}$ as

$$\bar{\underset{\sim}{\omega}} = D^* \underset{\sim}{\omega}. \qquad (3.443)$$

Equation (3.443) can be viewed as "averaging over the last subscript" in model (3.429).

Although $E(\underset{\sim}{\omega}) = \underset{\sim}{0}$ $\Big($see Lemma 3.13(a)$\Big)$, it is still convenient, for mathematical reasons, to make a transformation that results in a random vector having a "zero mean vector." This transformation is similar in spirit to that given in (3.420).

Let H_1^* represent the last $c_{s-1} - 1$ rows of the $c_{s-1} \times c_{s-1}$ Helmert matrix $\Big($see Searle (1982, p.71)$\Big)$. Then, similar to (3.419), H_1^* possesses the following properties:

$$\text{(i)} \quad H_1^* H_1^{*\prime} = I_{c_{s-1} - 1}.$$

$$\text{(ii)} \quad H_1^{*\prime} H_1^* = I_{c_{s-1}} - \bar{J}_{c_{s-1}}. \qquad (3.444)$$

$$\text{(iii)} \quad H_1^* \underset{\sim}{1}_{c_{s-1}} = \underset{\sim}{0}_{c_{s-1} - 1}.$$

Let $\underset{\sim}{u}^*$ be the $(c_{s-1}-1)\times 1$ random vector

$$\underset{\sim}{u}^* = H_1^*\underset{\sim}{\overline{\omega}} \qquad (3.445)$$

The next lemma establishes some statistical properties for $\underset{\sim}{u}^*$.

Lemma 3.15

The random vector $\underset{\sim}{u}^*$ is normally distributed with

(a) $E(\underset{\sim}{u}^*) = \underset{\sim}{0}_{c_{s-1}-1}$

(b) $\mathrm{Var}(\underset{\sim}{u}^*) = \sum_{i=1}^{s-2} H_1^*\tilde{B}_i H_1^{*\prime}\sigma_i^2 + \sigma_{s-1}^2 I_{c_{s-1}-1} + \delta H_1^*(A_{s-1}'A_{s-1})^{-1}H_1^{*\prime}$,

where

$$\begin{aligned} \tilde{B}_i &= I_{c_i} \otimes J_{\overline{c}_i}, \\ & \qquad\qquad i = 1,2,\cdots,s\text{-}2 \\ \overline{c}_i &= \prod_{j=i+1}^{s-1} a_j, \end{aligned} \qquad (3.446)$$

(Note that $c_i\overline{c}_i = c_{s-1}$, $i = 1,2,\cdots,s\text{-}2$, see (3.389).)

(c) $\underset{\sim}{u}^*$ is independent of $SSE_{\underset{\sim}{\omega}}$, where $SSE_{\underset{\sim}{\omega}}$ was given in (3.436).

Proof:

First note that since $\underset{\sim}{\omega}$ is normally distributed (see Lemma 3.13), $\underset{\sim}{\bar{\omega}}$ is normally distributed $\Big($see (3.443)$\Big)$, and hence $\underset{\sim}{u}^*$ is normally distributed.

(a) From (3.443), (3.445), and Lemma 3.13(a) we have,

$$E(\underset{\sim}{u}^*) = H_1^* E(\underset{\sim}{\bar{\omega}})$$

$$= H_1^* D^* E(\underset{\sim}{\omega}) = \underset{\sim}{0}_{c_{s-1}-1}.$$

(b) From (3.443), (3.445), and Lemma 3.13(b) we have

$$\text{Var}(\underset{\sim}{u}^*) = H_1^* D^* \left[\sum_{i=1}^{s-1} H_1 \tilde{A}_i H_1' \sigma_i^2 + \delta I_{m-1}\right] D^{*\prime} H_1^{*\prime}$$

$$= \sum_{i=1}^{s-1} H_1^* D^* H_1 \tilde{A}_i H_1' D^{*\prime} H_1^{*\prime} \sigma_i^2 + \delta H_1^* D^* D^{*\prime} H_1^{*\prime}. \qquad (3.447)$$

Now,

(i) $$H_1^* D^* D^{*\prime} H_1^{*\prime} = H_1^* (A_{s-1}' A_{s-1})^{-1} A_{s-1}' H_1' H_1 A_{s-1} (A_{s-1}' A_{s-1})^{-1} H_1^{*\prime}$$

$\Big($see (3.442)$\Big)$

$$= H_1^* \Big\{ (A_{s-1}' A_{s-1})^{-1} A_{s-1}' A_{s-1} (A_{s-1}' A_{s-1})^{-1}$$

$$- (A_{s-1}' A_{s-1})^{-1} A_{s-1}' \bar{J}_m A_{s-1} (A_{s-1}' A_{s-1})^{-1} \Big\} H_1^{*\prime}$$

$\Big($see (3.419)$\Big)$.

From (3.413),

$$(A'_{s-1}A_{s-1})^{-1}A'_{s-1}\bar{J}_m = \Big(\underset{\theta_{s-1}}{\oplus}\, m^{-1}_{\theta_{s-1}}\underset{\sim}{1}'_{m_{\theta_{s-1}}}\Big)\tfrac{1}{m}\underset{\sim}{1}_m\underset{\sim}{1}'_m$$

$$= \tfrac{1}{m}\underset{\sim}{1}_{c_{s-1}}\underset{\sim}{1}'_m .$$

Hence, from (3.444),

$$H_1^*(A'_{s-1}A_{s-1})^{-1}A'_{s-1}\bar{J}_m = \tfrac{1}{m}H_1^*\underset{\sim}{1}_{c_{s-1}}\underset{\sim}{1}'_m = \underset{\sim}{0}_{c_{s-1}-1} . \tag{3.448}$$

Thus,

$$H_1^*D^*D^{*\prime}H_1^{*\prime} = H_1^*(A'_{s-1}A_{s-1})^{-1}H_1^{*\prime} . \tag{3.449}$$

(ii) In the first term of (3.447) consider $H_1^*D^*H_1A_i$, $i = 1,2,\cdots,s-1$. We have, for $i = 1,2,\cdots,s-1$,

$$H_1^*D^*H_1A_i = H_1^*(A'_{s-1}A_{s-1})^{-1}A'_{s-1}H_1'H_1A_i \qquad \Big(\text{see } (3.442)\Big)$$

$$= H_1^*(A'_{s-1}A_{s-1})^{-1}A'_{s-1}A_i - H_1^*(A'_{s-1}A_{s-1})^{-1}A'_{s-1}\bar{J}_mA_i \qquad \Big(\text{see } (3.419)\Big)$$

$$= H_1^*(A'_{s-1}A_{s-1})^{-1}A'_{s-1}A_i . \qquad \Big(\text{see } (3.448)\Big)$$

Now, since range$(A_i) \subset$ range(A_j), $1 \le i < j \le s-1$ $\Big($see (3.416)$\Big)$, there exists matrices B_j of order $c_{s-1} \times c_j$, $1 \le j \le s-2$, such that

$$A_j = A_{s-1}B_j, \quad j = 1,2,\cdots,s-2. \tag{3.450}$$

Consequently, for $i = 1,2,\cdots,s-2$,

$$H_1^*(A'_{s-1}A_{s-1})^{-1}A'_{s-1}A_i = H_1^*(A'_{s-1}A_{s-1})^{-1}A'_{s-1}A_{s-1}B_i$$

$$= H_1^*B_i .$$

That is

$$H_1^*D^*H_1A_i = \begin{cases} H_1^*B_i, & i = 1,2,\cdots,s-2 \\ H_1^*, & i = s-1. \end{cases} \tag{3.451}$$

Therefore, from (3.449), (3.451), and (3.447) we have

$$\mathrm{Var}(\underset{\sim}{u}^*) = \sum_{i=1}^{s-1} H_1^*D^*H_1\tilde{A}_iH_1'D^{*\prime}H_1^{*\prime}\sigma_i^2 + \delta H_1^*D^*D^{*\prime}H_1^{*\prime}$$

$$= \sum_{i=1}^{s-2} H_1^*B_iB_i'H_1^{*\prime}\sigma_i^2 + H_1^*H_1^{*\prime}\sigma_{s-1}^2 + \delta H_1^*(A'_{s-1}A_{s-1})^{-1}H_1^{*\prime}$$

$$= \sum_{i=1}^{s-2} H_1^*\tilde{B}_iH_1^{*\prime}\sigma_i^2 + I_{c_{s-1}-1}\sigma_{s-1}^2 + \delta H_1^*(A'_{s-1}A_{s-1})^{-1}H_1^{*\prime},$$

where $\tilde{B}_i = B_iB_i'$, $i = 1,2,\cdots,s-1$. From (3.413) and the fact that A_{s-1} has full column rank, B_j can be (uniquely) expressed as

$$B_j = I_{c_j} \otimes \underset{\sim}{1}_{\bar{c}_j}, \quad j = 1,2,\cdots,s-2 \tag{3.452}$$

where

$$\bar{c}_j = \prod_{k=j+1}^{s-1} a_k, \quad j = 1, 2, \cdots, s-2.$$

Thus,

$$\tilde{B}_j = B_j B_j' = I_{c_j} \otimes J_{\bar{c}_j}, \quad j = 1, 2, \cdots, s-2.$$

(c) Since $\underset{\sim}{u}^* = H_1^* \bar{\underset{\sim}{\omega}}$, it suffices to show that $\bar{\underset{\sim}{\omega}}$ and $SSE_{\underset{\sim}{\omega}}$ are independent. From (3.442), (3.443), and (3.436), we have

$$(A_{s-1}'A_{s-1})^{-1}A_{s-1}'H_1'[\mathrm{Var}(\underset{\sim}{\omega})](I_{m-1} - Q_{s-1})$$

$$= \delta(A_{s-1}'A_{s-1})^{-1}A_{s-1}'H_1'(I_{m-1} - Q_{s-1}) \quad \Big(\text{see } (3.440)\Big).$$

However, since

$$\mathrm{range}(H_1 A_{s-1}) = \mathrm{range}(Q_{s-1}) \qquad \Big(\text{see } (4.333)\Big)$$

it follows that

$$A_{s-1}'H_1'(I_{m-1} - Q_{s-2}) = 0.$$

Thus $\bar{\underset{\sim}{\omega}}$ (and hence $\underset{\sim}{u}^*$) is independent of $SSE_{\underset{\sim}{\omega}}$. □

In the expression for Var($\underset{\sim}{u}^*$) given in Lemma 3.15(b), the matrices $H_1^*\tilde{B}_iH_1^{*\prime}$, $i = 1,2,\cdots,s-2$, are analogous to their counterparts in a balanced model in the sense that they can be shown to commute in pairs. This is the focal point of the next subsection.

The matrices $H_1^*\tilde{B}_iH_1^{*\prime}$, $i = 1,2,\cdots,s-2$. In this subsection we show that the matrices $H_1^*\tilde{B}_iH_1^{*\prime}$, $i = 1,2,\cdots,s-2$, commute in pairs (see Lemma 3.16). To do so, we first note that for $1 \leq i < j \leq s-2$,

$$\tilde{B}_i\tilde{B}_j = (I_{c_i} \otimes J_{\bar{c}_i})(I_{c_j} \otimes J_{\bar{c}_j}) \qquad \big(\text{see } (3.446)\big)$$

$$= (I_{a_1} \otimes I_{a_2} \otimes \cdots \otimes I_{a_i} \otimes J_{a_{i+1}} \otimes \cdots \otimes J_{a_j} \cdots \otimes J_{a_{s-1}})$$
$$\times (I_{a_1} \otimes I_{a_2} \otimes \cdots \otimes I_{a_i} \otimes I_{a_{i+1}} \otimes \cdots \otimes I_{a_j} \otimes J_{a_{j+1}} \otimes \cdots \otimes J_{a_{s-1}}) \qquad \big(\text{see } (3.389)\big)$$

$$= I_{c_i} \otimes J_{a_{i+1}} \otimes \cdots \otimes J_{a_j} \otimes a_{j+1}J_{a_{j+1}} \otimes \cdots \otimes a_{s-1}J_{a_{s-1}}$$

$$= \bar{c}_j(I_{c_i} \otimes J_{\bar{c}_i}). \tag{3.453}$$

Consequently, $\tilde{B}_i\tilde{B}_j$ is symmetric which implies that $\tilde{B}_i$ and $\tilde{B}_j$ commute.

Lemma 3.16

The matrices $H_1^*\tilde{B}_iH_1^{*\prime}$ commute in pairs for $i = 1,2,\cdots,s-2$.

Proof:

Without loss of generality take $1 \leq i < j \leq s-2$. Then,

$$(H_1^*\tilde{B}_iH_1^{*\prime})(H_1^*\tilde{B}_jH_1^{*\prime})$$

$$= H_1^*\tilde{B}_i(I_{c_{s-1}} - \bar{J}_{c_{s-1}})\tilde{B}_jH_1^{*\prime} \qquad \Big(\text{see } (3.444)\Big)$$

$$= H_1^*\tilde{B}_i\tilde{B}_jH_1^{*\prime} - H_1^*\tilde{B}_i\bar{J}_{c_{s-1}}\tilde{B}_jH_1^{*\prime}. \qquad (3.454)$$

Now, from (3.446),

$$\tilde{B}_i\bar{J}_{c_{s-1}} = (I_{c_i} \otimes J_{\bar{c}_i})\bar{J}_{c_{s-1}}$$

$$= \frac{1}{c_{s-1}}(I_{c_i} \otimes J_{\bar{c}_i})(J_{c_i} \otimes J_{\bar{c}_i}),$$

$$= \frac{1}{c_{s-1}}(J_{c_i} \otimes \bar{c}_iJ_{\bar{c}_i})$$

$$= \frac{\bar{c}_i}{c_{s-1}}J_{c_{s-1}},$$

recall that $c_i\bar{c}_i = c_{s-1}$, $i = 1,2,\cdots,s-1$ $\Big($see (3.389) and (3.446)$\Big)$.

Consequently,

$$H_1^*\tilde{B}_i\bar{J}_{c_{s-1}} = \frac{\bar{c}_i}{c_{s-1}}H_1^*J_{c_{s-1}} = 0. \qquad \Big(\text{see } (3.444)\Big)$$

Thus, the second term in (3.454) vanishes and hence

$$(H_1^*\tilde{B}_iH_1^{*\prime})(H_1^*\tilde{B}_jH_1^{*\prime}) = H_1^*\tilde{B}_i\tilde{B}_jH_1^{*\prime}.$$

Therefore, since $\tilde{B}_i\tilde{B}_j$ is symmetric $\big($see (3.452)$\big)$, it follows that $(H_1^*\tilde{B}_iH_1^{*\prime})(H_1^*\tilde{B}_jH_1^{*\prime})$ is symmetric and consequently $H_1^*\tilde{B}_iH_1^{*\prime}$ and $H_1^*\tilde{B}_jH_1^{*\prime}$ commute. □

This lemma indicates that there exists a $(c_{s-1}-1)\times(c_{s-1}-1)$ orthogonal matrix S which simultaneously diagonalizes $H_1^*\tilde{B}_1H_1^{*\prime},\cdots,H_1^*\tilde{B}_{s-2}H_1^{*\prime}$. Before we apply this transformation let us state some further properties concerning the $H_1^*\tilde{B}_iH_1^{*\prime}$, $i=1,2,\cdots,s-2$.

(i) The matrices B_i, $i=1,2,\cdots,s-2$, defined in (3.452), satisfy

$$\mathrm{range}(B_1)\subset\cdots\subset\mathrm{range}(B_{s-2}). \tag{3.455}$$

Thus, it follows that

$$\mathrm{range}(H_1^*B_1)\subset\cdots\subset\mathrm{range}(H_1^*B_{s-2}). \tag{3.456}$$

(ii) For $i=1,2,\cdots,s-2$,

$$\mathrm{rank}(H_1^*B_i)=\mathrm{rank}(H_1^*\tilde{B}_iH_1^{*\prime})=c_i-1. \tag{3.457}$$

To verify (3.455) let $1 \leq i < j \leq s-2$ and note that $\Big($see (3.389) and (3.446)$\Big)$,

$$\bar{c}_i = \left(\frac{c_j}{c_i}\right)\bar{c}_j .$$

Hence, from (3.452)

$$B_i = I_{c_i} \otimes \underline{1}_{\bar{c}_i} = I_{c_i} \otimes \underline{1}_{\left(\frac{c_j}{c_i}\right)\bar{c}_j}$$

$$= I_{c_i} \otimes \underline{1}_{\frac{c_j}{c_i}} \otimes \underline{1}_{\bar{c}_j}$$

$$= (I_{c_i} \otimes I_{\frac{c_j}{c_i}} \otimes \underline{1}_{\bar{c}_j})(I_{c_i} \otimes \underline{1}_{\frac{c_j}{c_i}})$$

$$= (I_{c_j} \otimes \underline{1}_{\bar{c}_j})(I_{c_i} \otimes \underline{1}_{\frac{c_j}{c_i}})$$

$$= B_j(I_{c_i} \otimes \underline{1}_{\frac{c_j}{c_i}}) .$$

Therefore, $\text{range}(B_i) \subset \text{range}(B_j)$, $1 \leq i < j \leq s-2$. Since, for $1 \leq i < j \leq s-2$,

$$H_1^* B_i = H_1^* B_j(I_{c_i} \otimes \underline{1}_{\frac{c_j}{c_i}}) ,$$

it follows that

$$\mathrm{range}(H_1^*B_i) \subset \mathrm{range}(H_1^*B_j), \quad 1 \le i < j \le s\text{-}2.$$

To verify (3.457) first note that

$$B_i \underset{\sim}{1}_{c_i} = \underset{\sim}{1}_{c_{s-1}}, \quad i = 1,2,\cdots,s\text{-}2$$

and

$$H_1^* \underset{\sim}{1}_{c_{s-1}} = \underset{\sim}{0}_{c_{s-1}-1} \qquad \big(\text{see } (3.444)\big).$$

Consequently, the c_i columns of $H_1^*B_i$ are linearly independent ⇒ $\mathrm{rank}(H_1^*B_i) \le c_i\text{-}1$, $i = 1,2,\cdots,s\text{-}2$. On the other hand, by the Frobenius Inequality (see Lemma A.1), for $i = 1,2,\cdots,s\text{-}2$

$$\begin{aligned}\mathrm{rank}(H_1^*B_i) &\ge \mathrm{rank}(H_1^*) + \mathrm{rank}(B_i) - \mathrm{rank}(I_{c_{s-1}}) \\ &= c_{s-1}\text{-}1 + c_i - c_{s-1} \\ &= c_i - 1.\end{aligned}$$

Thus, $\mathrm{rank}(H_1^*B_i) = \mathrm{rank}(H_1^*\tilde{B}_i H_1^{*\prime}) = c_i\text{-}1$, $i = 1,2,\cdots,s\text{-}2$.

As indicated earlier, Lemma 3.16 implies that there exists a $(c_{s-1} - 1) \times (c_{s-1} \times 1)$ orthogonal matrix S which simultaneously diagonalizes $H_1^*\tilde{B}_1H_1^{*\prime}, \cdots, H_1^*\tilde{B}_{s-2}H_1^{*\prime}$. The construction of S is our focus in the next subsection.

The construction of S. We now describe how to construct the $(c_{s-1}-1)\times(c_{s-1}-1)$ orthogonal matrix S. To begin with consider the matrix $B_i^{\prime}H_1^{*\prime}H_1^*B_i$, $i=1,2,\cdots,s\text{-}2$. From (3.446) and (3.452) and recalling that $c_i\bar{c}_i=c_{s-1}$, $i=1,2,\cdots,s\text{-}2$, we have

$$
\begin{aligned}
B_i^{\prime}H_1^{*\prime}H_1^*B_i &= B_i^{\prime}(I_{c_{s-1}}-\bar{J}_{c_{s-1}})B_i \\
&= (I_{c_i}\otimes \mathbf{1}_{\bar{c}_i}^{\prime})(I_{c_{s-1}}-\bar{J}_{c_{s-1}})(I_{c_i}\otimes \mathbf{1}_{\bar{c}_i}) \\
&= \left[(I_{c_i}\otimes \mathbf{1}_{\bar{c}_i}^{\prime})-\frac{1}{c_{s-1}}(I_{c_i}\otimes \mathbf{1}_{\bar{c}_i}^{\prime})(J_{c_i}\otimes J_{\bar{c}_i})\right](I_{c_i}\otimes \mathbf{1}_{\bar{c}_i}) \\
&= \left[(I_{c_i}\otimes \mathbf{1}_{\bar{c}_i}^{\prime})-\frac{\bar{c}_i}{c_{s-1}}(J_{c_i}\otimes \mathbf{1}_{\bar{c}_i}^{\prime})\right](I_{c_i}\otimes \mathbf{1}_{\bar{c}_i}) \\
&= \left((I_{c_i}-\bar{J}_{c_i})\otimes \mathbf{1}_{\bar{c}_i}^{\prime}\right)(I_{c_i}\otimes \mathbf{1}_{\bar{c}_i}) \\
&= \bar{c}_i(I_{c_i}-\bar{J}_{c_i}), \quad i=1,2,\cdots,s\text{-}2.
\end{aligned}
$$

Hence, a generalized inverse of $B_i^{\prime}H_1^{*\prime}H_1^*B_i$ is

$$
(B_i^{\prime}H_1^{*\prime}H_1^*B_i)^{-}=\frac{1}{\bar{c}_i}I_{c_i}, \quad i=1,2,\cdots,s\text{-}2.
$$

Consequently, the matrix $H_1^*B_i(B_i^{\prime}H_1^{*\prime}H_1^*B_i)^{-}B_i^{\prime}H_1^{*\prime}=\frac{1}{\bar{c}_i}H_1^*\tilde{B}_iH_1^{*\prime}$ is symmetric, idempotent, and has rank c_i-1, $i=1,2,\cdots,s\text{-}2$. Now, consider the identity

$$I_{c_{s-1}-1} = H_1^* H_1^{*\prime} \qquad \text{(see (3.444))}$$

$$= U_1 + U_2 + \cdots + U_{s-1},$$

where

$$U_i = \frac{1}{\bar{c}_i} H_1^* \tilde{B}_i H_1^{*\prime} - \frac{1}{\bar{c}_{i-1}} H_1^* \tilde{B}_{i-1} H_1^{*\prime}, \quad i = 1,2,\cdots,s\text{-}1.$$

(We take $\bar{c}_0 \equiv c_{s-1}$; $\tilde{B}_0 \equiv 0$; $\bar{c}_{s-1} \equiv 1$; $\tilde{B}_{s-1} \equiv I$.) From (3.456) and the fact that $\frac{1}{\bar{c}_i} H_1^* \tilde{B}_i H_1^{*\prime}$ is a projection matrix, $i = 1,2,\cdots,s\text{-}2$, it follows that U_i is a projection matrix and furthermore range(U_i) is the orthogonal complement of range$\left(\frac{1}{\bar{c}_{i-1}} H_1^* \tilde{B}_{i-1} H_1^{*\prime}\right)$, $i = 1,2,\cdots,s\text{-}1$ (see Christensen (1987, p. 339)). Also, for $c_0 \equiv 1$,

$$\text{rank}(U_i) = c_i - c_{i-1}, \quad i = 1,2,\cdots,s\text{-}1.$$

Let S_i be a $(c_i - c_{i-1}) \times (c_{s-1} - 1)$ matrix of rank $c_i - c_{i-1}$ whose rows of orthonormal and span the row space of U_i, $i = 1,2,\cdots,s\text{-}1$. Then similar to Khuri's (1990) Lemma 3.4 (ii), it is easy to verify that

$$S_i S_j' = \begin{cases} I_{c_i - c_{i-1}}, & i = j = 1,2,\cdots,s\text{-}1 \\ 0, & i \neq j. \end{cases}$$

Now, define the $(c_{s-1} - 1) \times (c_{s-1} - 1)$ matrix S as

$$S = [S_1' : S_2' : \cdots : S_{s-1}']'.$$

Then S is orthogonal and diagonalizes $H_1^*\tilde{B}_1H_1^{*\prime}, \cdots, H_1^*\tilde{B}_{s-2}H_1^{*\prime}$ simultaneously.

In the following subsection we consider the transformation $\underset{\sim}{x} = S\underset{\sim}{u}^*$.

The transformation $\underset{\sim}{x} = S\underset{\sim}{u}^*$. Define the $(c_{s-1}-1)\times 1$ vector $\underset{\sim}{x}$ as

$$\underset{\sim}{x} = S\underset{\sim}{u}^*$$

where S is the orthogonal matrix mentioned above and $\underset{\sim}{u}^*$ was defined in (3.445). From Lemma 3.15 we have

$$E(\underset{\sim}{x}) = \underset{\sim}{0}_{c_{s-1}-1}$$

$$(3.468)$$

$$\mathrm{Var}(\underset{\sim}{x}) = \sum_{i=1}^{s-2}\Delta_i\sigma_i^2 + \sigma_{s-1}^2 I_{c_{s-1}-1} + \delta S H_1^*(A_{s-1}^{\prime}A_{s-1})^{-1}H_1^{*\prime}S,$$

where $\Delta_i = SH_1^*\tilde{B}_iH_1^{*\prime}S^{\prime}$ is a diagonal matrix consisting of the eigenvalues of $H_1^*\tilde{B}_iH_1^{*\prime}$. The matrix Δ_i is of order $(c_{s-1}-1)$
$\times(c_{s-1}-1)$,
$i=1,2,\cdots,s\text{-}2$. Lemma 3.17 provides the form for Δ_i, $i=1,2,\cdots,s\text{-}2$.

Lemma 3.17

The eigenvalues of $H_1^*\tilde{B}_iH_1^{*\prime}$ are 0 with multiplicity $c_{s-1}-c_i$ and $\bar{c}_i$ with multiplicity c_i-1, $i=1,2,\cdots,s\text{-}2$, $\left(\bar{c}_i\right.$ is given in $\left.(3.446)\right)$. Furthermore, we can express Δ_i as

$$\Delta_i = \bar{c}_i \begin{pmatrix} I_{c_i-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad i = 1,2,\cdots,s\text{-}2. \tag{3.459}$$

Proof:

Since $H_1^* \tilde{B}_i H_1^{*\prime}$ is of order $(c_{s-1}-1) \times (c_{s-1}-1)$ and has rank $c_i - 1$ $\big($see (3.457)$\big)$, 0 is an eigenvalue with multiplicity $c_{s-1} - 1 - (c_i - 1) = c_{s-1} - c_i$, $i = 1,2,\cdots,s\text{-}2$. Also, the nonzero eigenvalues of $H_1^* \tilde{B}_i H_1^{*\prime}$ are equivalent to the nonzero eigenvalues of $\tilde{B}_i H_1^{*\prime} H_1^*$, $i = 1,2,\cdots,s\text{-}2$. However, for $i = 1,2,\cdots,s\text{-}2$

$$\tilde{B}_i H_1^{*\prime} H_1^* = (I_{c_i} \otimes J_{\bar{c}_i})(I_{c_{s-1}} - \bar{J}_{c_{s-1}}) \ \big(\text{see (3.444) and (3.446)}\big).$$

Recalling that $c_i \bar{c}_i = c_{s-1}$ for $i = 1,2,\cdots,s\text{-}2$ $\big($see (3.446) and (3.389)$\big)$, we have

$$\frac{1}{c_{s-1}} J_{c_{s-1}} = \frac{1}{c_{s-1}} [J_{c_i} \otimes J_{\bar{c}_i}].$$

Thus, for $i = 1,2,\cdots,s\text{-}2$,

$$\tilde{B}_i H_1^{*\prime} H_1^* = (I_{c_i} \otimes J_{\bar{c}_i}) - \frac{1}{c_{s-1}} (I_{c_i} \otimes J_{\bar{c}_i})(J_{c_i} \otimes J_{\bar{c}_i})$$

$$= (I_{c_i} \otimes J_{\bar{c}_i}) - \frac{\bar{c}_i}{c_{s-1}} (J_{c_i} \otimes J_{\bar{c}_i})$$

$$= (I_{c_i} \otimes J_{\bar{c}_i}) - (\bar{J}_{c_i} \otimes J_{\bar{c}_i}) \quad (\text{recall, } c_i \bar{c}_i = c_{s-1})$$

$$= (I_{c_i} - \bar{J}_{c_i}) \otimes J_{\bar{c}_i}.$$

Now, the nonzero eigenvalues of $(I_{c_i} - \bar{J}_{c_i}) \otimes J_{\bar{c}_i}$ are the products of the nonzero eigenvalues of $(I_{c_i} - \bar{J}_{c_i})$ with those of $J_{\bar{c}_i}$. Thus, since $I_{c_i} - \bar{J}_{c_i}$ is idempotent with rank $c_i - 1$ and the only nonzero eigenvalue of $J_{\bar{c}_i}$ is $\bar{c}_i$, the nonzero eigenvalues of $\tilde{B}_i H_i^{*\prime} H_i^*$ are $\bar{c}_i$ with multiplicity $c_i - 1$, $i = 1,2,\cdots,s-2$.

Since the diagonal elements of Δ_i are the eigenvalues of $H_i^* \tilde{B}_i H_i^{*\prime}$ and since (3.456) implies that $\text{range}(\Delta_1) \subset \text{range}(\Delta_2) \subset \cdots \subset \text{range}(\Delta_{s-2})$ which in turn implies that $\text{rank}(\Delta_1 + \Delta_2 + \cdots + \Delta_j) = \text{rank}(\Delta_j) = c_j - 1$, $j = 1,2,\cdots,s-2$, we can, without loss of generality express Δ_i as

$$\Delta_i = \bar{c}_i \begin{pmatrix} I_{c_i-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad i = 1,2,\cdots,s-2. \qquad \square$$

In (3.458) the variance-covariance matrix of $\underset{\sim}{x}$ is not diagonal. To diagonal this matrix we make a transformation utilizing resampling. This is done in the next subsection.

Utilizing resampling. Consider once again $SSE_{\underset{\sim}{\omega}}$ defined in (3.346). Let R^* be the $(m-1) \times (m-1)$ symmetric, idempotent matrix

$$R^* = I_{m-1} - Q_{s-1}. \tag{3.460}$$

$\big(Q_j$, $j = 1,2,\cdots,s-1$, is defined in (3.433)$\big)$. Then

$$SSE_{\underset{\sim}{\omega}} = \underset{\sim}{\omega} R^* \underset{\sim}{\omega}.$$

Clearly R^* has rank $m - c_{s-1}$. Now, we can express R^* as

$$R^* = C^*\Lambda^*C^{*\prime} \tag{3.461}$$

where

$$C^* = [C_1^*:C_2^*:C_3^*]$$
$$\Lambda^* = \text{diag}(I_{\eta_1^*}, I_{\eta_2^*}, 0) \tag{3.462}$$

$$\eta_1^* = c_{s-1} - 1$$
$$\eta_2^* = m - 2c_{s-1} + 1 \tag{3.463}$$

and C_1^*, C_2^*, and C_3^* are of orders $m \times \eta_1^*$, $m \times \eta_2^*$, and $m \times c_{s-1}$, respectively. We note that $\eta_1^* + \eta_2^* = m - c_{s-1} = \text{rank}(R^*)$. Also, $\eta_2^* > 0$ by (3.393).

In order to diagonal the variance-covariance matrix of $\underset{\sim}{x}$ $\Big($see (3.458)$\Big)$ we introduce the following transformation

$$\underset{\sim}{\tau} = \underset{\sim}{x} + (\lambda_{\max}^* I_{\eta_1^*} - L^*)^{\frac{1}{2}} C_1^{*\prime} \underset{\sim}{\omega}, \tag{3.464}$$

where $L^* = SH_1^*(A_{s-1}'A_{s-1})^{-1}H_1^{*\prime}S'$ and $\lambda_{\max}^*$ is its largest eigenvalue. The vector $\underset{\sim}{x}$ was defined in (3.457) and C_1^* in (3.462). The random vector $\underset{\sim}{\omega}$ was given in (3.429). The matrix $(\lambda_{\max}^* I_{\eta_1^*} - L^*)^{\frac{1}{2}}$ is well-defined since $\lambda_{\max}^* I_{\eta_1^*} - L^*$ is p.s.d. It's eigenvalues are the square roots of the eigenvalues of $\lambda_{\max}^* I_{\eta_1^*} - L^*$.

Theorem 3.14.

The random vector $\underset{\sim}{\tau}$ is normally distributed with

(a) $E(\underset{\sim}{\tau}) = \underset{\sim}{0}_{c_{s-1}-1}$

$$\tag{3.465}$$

(b) $\mathrm{Var}(\underset{\sim}{\tau}) = \sum_{i=1}^{s-2} \Delta_i \sigma_i^2 + \delta^* I_{c_{s-1}-1}$

where $\delta^* = \sigma_{s-1}^2 + \lambda_{\max}^* \delta$ and δ was defined in Lemma 3.13(b).

Proof:

The proof of Theorem 3.14 follows similarly to that of Lemma 3.13 and is therefore omitted. □

The exact tests. Let us partition $\underset{\sim}{\tau}$ as

$$\underset{\sim}{\tau} = (\underset{\sim}{\tau}_1' : \underset{\sim}{\tau}_2' : \cdots : \underset{\sim}{\tau}_{s-1}')' \tag{3.466}$$

where $\underset{\sim}{\tau}_i$ is of order $(c_i - c_{i-1}) \times 1$, $i = 1, 2, \cdots, s-1$ $(c_0 \equiv 1)$. Note that $\sum_{i=1}^{s-1} (c_i - c_{i-1}) = c_{s-1} - 1$. Then the following facts are apparent:

(i) $\underset{\sim}{\tau}_i \sim N\Big(\underset{\sim}{0},\ (\sum_{j=i}^{s-2} \sigma_j^2 \bar{c}_j + \delta^*) I_{c_i - c_{i-1}}\Big)$, $i = 1, 2, \cdots, s-2$

$$\tag{3.467}$$

$\underset{\sim}{\tau}_{s-1} \sim N(\underset{\sim}{0},\ \delta^* I_{c_{s-1} - c_{s-2}})$.

This follows since we can use Lemma 3.17 to express $\mathrm{Var}(\underset{\sim}{\tau})$ in Theorem 3.14(b) as

$$\mathrm{Var}(\underset{\sim}{\tau}) = \mathrm{diag}\Big((\sum_{j=1}^{s-2} \sigma_j^2 \bar{c}_j + \delta^*) I_{c_1 - 1},\ (\sum_{j=2}^{s-2} \sigma_j^2 \bar{c}_j + \delta^*) I_{c_2 - c_1}, \cdots,$$

$$(\sum_{j=i}^{s-2} \sigma_j^2 \bar{c}_j + \delta^*) I_{c_i - c_{i-1}}, \cdots, (\sigma_{s-2}^2 \bar{c}_{s-2} + \delta^*) I_{c_{s-2} - c_{s-3}}, \delta^* I_{c_{s-1} - c_{s-2}} \Big).$$

(ii) $\underset{\sim}{\tau}_1, \underset{\sim}{\tau}_2, \cdots, \underset{\sim}{\tau}_{s-1}$ are mutually independent.

$$\text{(iii)}\quad \underset{\sim}{\tau}_i' \underset{\sim}{\tau}_i \Big/ \Big(\sum_{j=i}^{s-2} \sigma_j^2 \bar{c}_j + \delta^* \Big) \sim \chi^2_{c_i - c_{i-1}},\quad i = 1,2,\cdots,s-2$$

$$\underset{\sim}{\tau}_{s-1}' \underset{\sim}{\tau}_{s-1} / \delta^* \sim \chi^2_{c_{s-1} - c_{s-2}}. \tag{3.468}$$

Thus, if we wish to test $H_0: \sigma_i^2 = 0$ vs. $H_a: \sigma_i^2 \neq 0$, $i = 1,2,\cdots,s-2$, an appropriate test statistic would be

$$F_i = \frac{\underset{\sim}{\tau}_i' \underset{\sim}{\tau}_i / (c_i - c_{i-1})}{\underset{\sim}{\tau}_{i+1}' \underset{\sim}{\tau}_{i+1} / (c_{i+1} - c_i)},\quad i = 1,2,\cdots,s-2. \tag{3.469}$$

Under H_0, $F_i \sim F_{c_i - c_{i-1}, c_{i+1} - c_i}$, $i = 1,2,\cdots,s-2$. Under H_a, $F_i \sim (1 + \bar{c}_i \zeta_i) F_{c_i - c_{i-1}, c_{i+1} - c_i}$ where

$$\zeta_i = \frac{\sigma_i^2}{\sum_{j=i+1}^{s-2} \sigma_j^2 \bar{c}_j + \delta^*}, \tag{3.470}$$

$i = 1,2,\cdots,s-2$. Since $\bar{c}_i > 0$ for all $i = 1,2,\cdots,s-2$, we would reject H_0 for large values of our test statistic, F_i $(i = 1,2,\cdots,s-2)$.

3.4.6. Reduction of the Exact Tests when the Design is Balanced

Introduction. In this section we show that the exact tests reduce to the usual ANOVA tests when the design is balanced. For testing $H_0:\sigma_s^2 = 0$ vs. $H_a:\sigma_s^2 \neq 0$ the test statistic F_s, in (3.407), clearly reduces to the usual ANOVA test since it is based on the same mean squares as are used in a balanced data situation. In the remainder of this section then, we concentrate on showing that the exact tests developed for testing $H_0:\sigma_i^2 = 0$ vs. $H_a:\sigma_i^2 \neq 0$, $i = 1,2,\cdots,s-1$, reduce to their ANOVA counterparts when the design is balanced.

Reduction of the exact test for $H_0:\sigma_{s-1}^2 = 0$. The test statistic derived for testing $H_0:\sigma_{s-1}^2 = 0$ vs. $H_a:\sigma_{s-1}^2 \neq 0$ was defined in (3.441). It was based on the random vector $\underset{\sim}{\omega}$ defined in (3.426). Now, when the design is balanced, that is, when

$$m_{\theta_{s-1}} \equiv a_s$$

and (3.471)

$$n_{\theta_s} \equiv a_{s+1},$$

the matrix K $\left(\text{see } (3.418)\right)$ becomes

$$K = \frac{1}{a_s} I_{c_s}.$$

Hence L (see (4.427)) reduces to

$$L = \frac{1}{a_s} H_1 H_1' = \frac{1}{a_s} I_{c_s - 1}.$$

(In this case $c_s \equiv m = \prod_{i=1}^{s} a_i$ (see (3.389))).

Thus $\lambda_{\max}$, the largest eigenvalue of L, is $\frac{1}{a_s}$. Consequently $\underset{\sim}{\omega}$ (in (3.426)) simplifies to

$$\underset{\sim}{\omega} = \underset{\sim}{u} = H_1 \underset{\sim}{\bar{y}} \qquad \text{(see (3.420))}.$$

From (3.441) we can express the test statistic as

$$F_{s-1} = \frac{\underset{\sim}{\omega}'(Q_{s-1} - Q_{s-2})\underset{\sim}{\omega}/(c_{s-1} - c_{s-2})}{SSE_{\underset{\sim}{\omega}}/(m - c_{s-1})}$$

$$= \frac{\underset{\sim}{\bar{y}}' H_1' (Q_{s-1} - Q_{s-2}) H_1 \underset{\sim}{\bar{y}}/(c_{s-1} - c_{s-2})}{\underset{\sim}{\bar{y}}' H_1' (I_{m-1} - Q_{s-1}) H_1 \underset{\sim}{\bar{y}}/(m - c_{s-1})} \quad \text{(see (3.426))}. \tag{3.472}$$

Now, from (3.433) and (3.413), when the design is balanced we have for $j = 1, 2, \cdots, s-1$,

$$Q_j = H_1 A_j (A_j' A_j)^{-1} A_j' H_1'$$

$$= \frac{1}{\bar{c}_j a_s} H_1 [I_{c_j} \otimes J_{\bar{c}_j a_s}] H_1'. \qquad \text{(recall (3.446) and (3.471))}$$

Also, from (3.419),

$$H_1'H_1 = I_m - \bar{J}_m = I_{c_s} - \bar{J}_{c_s} \qquad (c_s \equiv m \text{ when the design is balanced})$$

$$= (I_{c_j} \otimes I_{\bar{c}_j a_s}) - \frac{1}{c_s}(J_{c_j} \otimes J_{\bar{c}_j a_s}).$$

Consequently for $j = 1, 2, \cdots, s-1$,

$$H_1'Q_jH_1 = \left[(I_{c_j} \otimes I_{\bar{c}_j a_s}) - (\bar{J}_{c_j} \otimes \bar{J}_{\bar{c}_j a_s})\right]\left[I_{c_j} \otimes \bar{J}_{\bar{c}_j a_s}\right]$$

$$\times\left[(I_{c_j} \otimes I_{\bar{c}_j a_s}) - (\bar{J}_{c_j} \otimes \bar{J}_{\bar{c}_j a_s})\right]$$

$$= \left[(I_{c_j} \otimes \bar{J}_{\bar{c}_j a_s}) - (\bar{J}_{c_j} \otimes \bar{J}_{\bar{c}_j a_s})\right]\left[(I_{c_j} \otimes I_{\bar{c}_j a_s}) - (\bar{J}_{c_j} \otimes \bar{J}_{\bar{c}_j a_s})\right]$$

$$= \left((I_{c_j} - \bar{J}_{c_j}) \otimes \bar{J}_{\bar{c}_j a_s}\right) \qquad \left((I_{c_j} - \bar{J}_{c_j})\bar{J}_{c_j} = 0\right). \qquad (3.473)$$

Thus, when the design is balanced, $H_1'(Q_{s-1} - Q_{s-2})H_1$ in the numerator of (3.472) becomes

$$\left[(I_{c_{s-1}} - \bar{J}_{c_{s-1}}) \otimes \bar{J}_{\bar{c}_{s-1}a_s}\right] - \left[(I_{c_{s-2}} - \bar{J}_{c_{s-2}}) \otimes \bar{J}_{\bar{c}_{s-2}a_s}\right]$$

$$= \left[\left((I_{c_{s-2}} \otimes I_{a_{s-1}}) - (\bar{J}_{c_{s-2}} \otimes \bar{J}_{a_{s-1}})\right) \otimes \bar{J}_{\bar{c}_{s-1}a_s}\right]$$

$$- \left[(I_{c_{s-2}} - \bar{J}_{c_{s-2}}) \otimes \bar{J}_{\bar{c}_{s-2}a_s}\right]$$

$$= (I_{c_{s-2}} \otimes I_{a_{s-1}} \otimes \bar{J}_{a_s}) - (\bar{J}_{c_{s-2}} \otimes \bar{J}_{a_{s-1}} \otimes \bar{J}_{a_s})$$

$$-\left[(I_{c_{s-2}} - \bar{J}_{c_{s-2}}) \otimes \bar{J}_{a_{s-1}} \otimes \bar{J}_{a_s}\right]$$

(recall that $\bar{c}_{s-2} = a_{s-1}$ and $\bar{c}_{s-1} \equiv 1$). Hence the numerator of F_{s-1} reduces to, finally

$$\underset{\sim}{\bar{y}}'\left[I_{c_{s-2}} \otimes (I_{a_{s-1}} - \bar{J}_{a_{s-1}}) \otimes \bar{J}_{a_s}\right]\underset{\sim}{\bar{y}}/(c_{s-1} - c_{s-2}). \tag{3.474}$$

Similarly, we can show that $H_1'(I_{m-1} - Q_{s-1})H_1$ reduces to

$$H_1'(I_{m-1} - Q_{s-1})H_1 = I_{c_{s-1}} \otimes (I_{a_s} - \bar{J}_{a_s}) \tag{3.475}$$

when the design is balanced (recall that $c_s \equiv m$).

In a balanced data situation, $\underset{\sim}{\bar{y}}$ $\Big($see (3.411) and (3.412)$\Big)$ becomes

$$\underset{\sim}{\bar{y}} = \frac{1}{a_{s+1}}(I_{c_s} \otimes \underset{\sim}{1}'_{a_{s+1}})\underset{\sim}{y}. \tag{3.476}$$

Therefore the numerator and denominator of (3.472) become, respectively, aside from a constant,

$$\frac{1}{a_s}\underset{\sim}{y}'\left[I_{c_{s-2}} \otimes (I_{a_{s-1}} - \bar{J}_{a_{s-1}}) \otimes \bar{J}_{a_s} \otimes \bar{J}_{a_{s+1}}\right]\underset{\sim}{y} \tag{3.477}$$

$$\frac{1}{a_s}\underset{\sim}{y}'\left[I_{c_{s-1}} \otimes (I_{a_s} - \bar{J}_{a_s}) \otimes \bar{J}_{a_{s+1}}\right]\underset{\sim}{y}. \tag{3.478}$$

Hence $a_s\underset{\sim}{\omega}'(Q_{s-1}-Q_{s-2})\underset{\sim}{\omega}$ and $a_s SSE_{\underset{\sim}{\omega}}$ reduce to the usual ANOVA sums of squares from a balanced model $\Big($see Khuri (1982)$\Big)$. Thus F_{s-1} $\Big($in (3.472)$\Big)$ reduces to the usual ANOVA F-test associated with the hypothesis $H_0:\sigma^2_{s-1}=0$.

Reduction of the exact tests for $H_0:\sigma^2_i=0$, $i=1,2,\cdots,s-2$. When testing $H_0:\sigma^2_i=0$, $j=1,2,\cdots,s-2$, we concerned ourselves with the vector $\underset{\sim}{\tau}$ defined in (3.464). When the design is balanced, the matrix L^* (given in the expression for $\underset{\sim}{\tau}$) reduces to

$$L^* = SH_1^*(A'_{s-1}A_{s-1})^{-1}H_1^{*\prime}S'$$

$$= SH_1^*(\tfrac{1}{a_s}I_{c_s})H_1^{*\prime}S' \qquad \Big(\text{see } (3.413) \text{ and } (3.471)\Big)$$

$$= \tfrac{1}{a_s}SS' \qquad \Big(\text{see } (3.444)\Big)$$

$$= \tfrac{1}{a_s}I_{c_{s-1}-1}. \qquad (S \text{ is orthogonal})$$

Consequently, its largest eigenvalue, $\lambda^*_{\max}$, is $\frac{1}{a_s}$. Therefore, when the design is balanced, $\underset{\sim}{\tau}$ becomes

$$\underset{\sim}{\tau} = \underset{\sim}{x} = S\underset{\sim}{u}^* \qquad \Big(\text{see } (3.457)\Big)$$

$$= SH_1^*\underset{\sim}{\bar{\omega}} \qquad \Big(\text{see } (3.445)\Big)$$

$$= SH_1^*D^*\underset{\sim}{\omega} \qquad \Big(\text{see } (3.446)\Big).$$

Hence,

$$\underset{\sim}{\tau}'\underset{\sim}{\tau} = \underset{\sim}{\omega}'D^{*\prime}H_1^{*\prime}S'SH_1^*D^*\underset{\sim}{\omega}$$

$$= \underset{\sim}{\omega}'D^{*\prime}(I_{c_{s-1}} - \bar{J}_{c_{s-1}})D^*\underset{\sim}{\omega} \qquad \Big(\text{see } (3.444)\Big)$$

$$= \underset{\sim}{\omega}'D^{*\prime}D^*\underset{\sim}{\omega} - \underset{\sim}{\omega}'D^{*\prime}\bar{J}_{c_{s-1}}D^*\underset{\sim}{\omega}$$

$$= \underset{\sim}{\omega}'\Big[\frac{1}{a_s}H_1(I_{c_{s-1}} \otimes \bar{J}_{a_s})H_1'\Big]\underset{\sim}{\omega}$$

$$- \underset{\sim}{\omega}'\Big[\frac{1}{a_s}H_1(\bar{J}_{c_{s-1}} \otimes \bar{J}_{a_s})H_1'\Big]\underset{\sim}{\omega} \qquad \Big(\text{see } (3.442) \text{ and } (3.413)\Big)$$

$$= \frac{1}{a_s}\underset{\sim}{\omega}'H_1\Big[(I_{c_{s-1}} - \bar{J}_{c_{s-1}}) \otimes \bar{J}_{a_s}\Big]H_1'\underset{\sim}{\omega}. \qquad (3.479)$$

As mentioned previously, when the design is balanced, $\underset{\sim}{\omega}$ reduces to

$$\underset{\sim}{\omega} = H_1\bar{\underset{\sim}{y}} = \frac{1}{a_{s+1}}H_1(I_{c_s} \otimes \underset{\sim}{1}'_{a_{s+1}})\underset{\sim}{y}.$$

Consequently, after some simplification, we can express (3.479) as

$$\underset{\sim}{\tau}'\underset{\sim}{\tau} = \frac{1}{a_s a_{s+1}}\underset{\sim}{y}'\Big[(I_{c_{s-1}} - \bar{J}_{c_{s-1}}) \otimes \bar{J}_{a_s} \otimes \bar{J}_{a_{s+1}}\Big]\underset{\sim}{y}. \qquad (3.480)$$

Now, recalling (3.398), when the design is balanced we have, for $j = 1,2,\cdots,s$,

$$P_j = X_j(X_j'X_j)^{-1}X_j'$$

$$= I_{c_j} \otimes \bar{J}_{\bar{n}_j} \qquad \left(\text{see } (3.384) \text{ and } (3.471)\right),$$

where

$$\bar{n}_j = n/c_j = \prod_{i=j+1}^{s+1} a_i, \quad j = 1, 2, \cdots, s$$

$$n = \prod_{i=1}^{s+1} a_i .$$

Therefore (3.480) can be written as (letting $P_0 \equiv \bar{J}_n$)

$$\underset{\sim}{\tau}'\underset{\sim}{\tau} = \frac{1}{a_s a_{s+1}} \underset{\sim}{y}'\left[(I_{c_{s-1}} \otimes \bar{J}_{a_s} \otimes \bar{J}_{a_{s+1}}) - (\bar{J}_{c_{s-1}} \otimes \bar{J}_{a_s} \otimes \bar{J}_{a_{s+1}})\right]\underset{\sim}{y}$$

$$= \frac{1}{a_s a_{s+1}} \underset{\sim}{y}'[P_{s-1} - P_0]\underset{\sim}{y}. \qquad (3.481)$$

However,

$$\sum_{j=1}^{s-1} (P_j - P_{j-1}) = P_{s-1} - P_0,$$

therefore

$$\underset{\sim}{\tau}'\underset{\sim}{\tau} = \frac{1}{a_s a_{s+1}} \underset{\sim}{y}'\left(\sum_{j=1}^{s-1} (P_j - P_{j-1})\right)\underset{\sim}{y}$$

$$= \frac{1}{a_s a_{s+1}} \sum_{j=1}^{s-1} \underset{\sim}{y}'(P_j - P_{j-1})\underset{\sim}{y}.$$

From Khuri (1982), the sums of squares $\underset{\sim}{y}'(P_j - P_{j-1})\underset{\sim}{y}$, $j = 1, 2, \cdots, s-1$, are the usual ANOVA sums of squares from a balanced model associated with the effects $\underset{\sim}{\beta}_1, \cdots, \underset{\sim}{\beta}_{s-1}$ (note that our "$P_j - P_{j-1}$" is Khuri's "P_j"). Thus $a_s a_{s+1} \underset{\sim}{\tau}'_j \underset{\sim}{\tau}_j$ reduces to the usual ANOVA sum of squares, $j = 1,2,\cdots,s-1$. Consequently, the test statistic developed to test $H_0: \sigma_i^2 = 0$, $i = 1,2,\cdots,s-2$, (see (3.469)) reduces to the ANOVA F-test when the design is balanced.

Expressions for the power of the exact tests are developed in the next section.

3.4.7. The Power of the Exact Tests

Introduction. We now investigate the power of the exact tests. We first concentrate on developing an expression for the power of the test associated with σ_i^2, $i = 1,2,\cdots,s-2$. Since the imbalance affects the last two stages only, our expression for this power function will be exact.

The approximate power of the test associated with σ_{s-1}^2 is derived using Hirotsu's (1979) procedure (see Appendix D). A similar expression can also be developed for the power of the test associated with σ_s^2.

The power of the exact test associated with $H_0: \sigma_i^2 = 0$, $i = 1,2,\cdots,s-2$. In testing $H_0: \sigma_i^2 = 0$ vs. $H_a: \sigma_i^2 \neq 0$, the appropriate test statistic was given in (3.469). If Ψ_i denotes the power associated with this test, then, for $i = 1,2,\cdots,s-2$,

$$\Psi_i = P(F_i > F_{\alpha, c_i - c_{i-1}, c_{i+1} - c_i} \mid H_a) \tag{3.482}$$

$$= P\left(\frac{\underline{z}_i'\underline{z}_i/(c_i - c_{i-1})}{\underline{z}_{i+1}'\underline{z}_{i+1}/(c_{i+1} - c_i)} > F_{\alpha, c_i - c_{i-1}, c_{i+1} - c_i} \,|\, H_a\right).$$

Under H_a, $F_i \sim (1+\bar{c}_i\zeta_i)F_{c_i - c_{i-1}, c_{i+1} - c_i}$ where $\bar{c}_i$ was defined in (3.446) and ζ_i in (3.470), $i = 1,2,\cdots,s\text{-}2$. Consequently, (3.482) becomes

$$\Psi_i = P\left(F_{c_i - c_{i-1}, c_{i+1} - c_i} > \frac{1}{1+\bar{c}_i\zeta_i} F_{\alpha, c_i - c_{i-1}, c_{i+1} - c_i}\right), \qquad (3.482)$$

$(i = 1,2,\cdots,s\text{-}2)$. Since Ψ_i is a monotone increasing function of ζ_i, it is a monotone decreasing function of δ^* (everything else being fixed) where δ^* was defined in Theorem 3.14(b). (Recall that

$$\zeta_i = \frac{\sigma_i^2}{\sum_{j=i+1}^{s-1} \sigma_j^2 \bar{c}_j + \delta^*}, \qquad i = 1,2,\cdots,s\text{-}2,$$

(see (3.470)).) From the definition of δ^* in Theorem 3.14(b) we have

$$\delta^* = \sigma_{s-1}^2 + \lambda_{\max}^* \delta$$

$$= \sigma_{s-1}^2 + \lambda_{\max}^*(\sigma_s^2 + \lambda_{\max}\sigma_\epsilon^2) \qquad \text{(see Lemma 3.13(b))}.$$

It is thus desirable for $\lambda_{\max}^*$ and $\lambda_{\max}^*\lambda_{\max}$ to be small. A sufficient condition for this to occur is when both $\lambda_{\max}^*$ and $\lambda_{\max}$ are small. The next lemma provides bounds for $\lambda_{\max}^*$ and $\lambda_{\max}$.

Lemma 3.18

$\lambda^*_{\max}$ and $\lambda_{\max}$ satisfy the double inequalities:

(a) $\frac{1}{c_{s-1}} \sum_{\theta_{s-1}} \frac{1}{m_{\theta_{s-1}}} \le \lambda^*_{\max} \le \frac{1}{m^{(1)}}$

(b) $\frac{1}{m} \sum_{\theta_s} \frac{1}{n_{\theta_s}} \le \lambda_{\max} \le \frac{1}{n^{(1)}}$, where

$$n^{(1)} = \min_{\theta_s} n_{\theta_s}$$

$$m^{(1)} = \min_{\theta_{s-1}} m_{\theta_{s-1}}.$$

Proof:

(a) $\lambda^*_{\max}$ is the largest eigenvalue of the $(c_{s-1}-1)\times(c_{s-1}-1)$ matrix $L^* = SH_1^*(A'_{s-1}A_{s-1})^{-1}H_1^{*\prime}S'$ (see (3.464)). Thus

$$\lambda^*_{\max} = e_{\max}\left(SH_1^*(A'_{s-1}A_{s-1})^{-1}H_1^{*\prime}S'\right)$$

$$= e_{\max}\left(H_1^*(A'_{s-1}A_{s-1})^{-1}H_1^{*\prime}\right) \qquad \text{(S is orthogonal)}$$

$$\le e_{\max}\left[(A'_{s-1}A_{s-1})^{-1}\right]e_{\max}(H_1^{*\prime}H_1^*)$$

$$= e_{\max}\left[(A'_{s-1}A_{s-1})^{-1}\right] \qquad (H_1^* \text{ has orthonormal rows})$$

$$= e_{\max}\left(\bigoplus_{\theta_{s-1}} m_{\theta_{s-1}}^{-1}\right) \qquad \left(\text{see } (3.413)\right)$$

$$= \frac{1}{m^{(1)}}, \text{ where } m^{(1)} = \min_{\theta_{s-1}} m_{\theta_{s-1}} .$$

It is also true that

$$\lambda^*_{\max} \geq \frac{1}{c_{s-1}-1} \mathrm{tr}\left[S H_1^* (A'_{s-1}A_{s-1})^{-1} H_1^{*\prime} S'\right],$$

the term on the right being the average of the eigenvalues of L*. Consequently,

$$\lambda^*_{\max} \geq \frac{1}{c_{s-1}-1} \mathrm{tr}\left[(A'_{s-1}A_{s-1})^{-1} H_1^{*\prime} H_1^*\right]$$

$$= \frac{1}{c_{s-1}-1} \mathrm{tr}\left[(A'_{s-1}A_{s-1})^{-1} - (A'_{s-1}A_{s-1})^{-1} \bar{J}_{c_{s-1}}\right] \qquad \left(\text{see } (3.444)\right)$$

$$= \frac{1}{c_{s-1}-1} \left\{ \mathrm{tr}\left[(A'_{s-1}A_{s-1})^{-1}\right] - \frac{1}{c_{s-1}} \underset{\sim}{1}'_{c_{s-1}} (A'_{s-1}A_{s-1}) \underset{\sim}{1}_{c_{s-1}} \right\}$$

$$= \frac{1}{c_{s-1}-1} \left[\sum_{\theta_{s-1}} m^{-1}_{\theta_{s-1}} - \frac{1}{c_{s-1}} \sum_{\theta_{s-1}} m^{-1}_{\theta_{s-1}} \right] \qquad \left(\text{see } (3.413)\right)$$

$$= \frac{1}{c_{s-1}} \sum_{\theta_{s-1}} m^{-1}_{\theta_{s-1}} .$$

Hence, the result follows. The proof of (b) is similar. □

It is more important that $\lambda^*_{\max}$ be small than for $\lambda_{\max}$ to be small. Even if $\lambda_{\max} = 1$ (it's maximum possible value regardless of (nested) design), the product $\lambda^*_{\max}\lambda_{\max}$ will be small if $\lambda^*_{\max}$ is small.

The power of the exact test associated with $H_0:\sigma^2_{s-1}=0$. We now obtain an expression for the power of the test for $H_0:\sigma^2_{s-1}=0$. In this case the appropriate test statistic is $\left(\text{see } (3.441)\right)$

$$F_{s-1}=\frac{\underset{\sim}{\omega}'(Q_{s-1}-Q_{s-2})\underset{\sim}{\omega}/(c_{s-1}-c_{s-2})}{SSE_{\underset{\sim}{\omega}}/(m-c_{s-1})},$$

where Q_j was defined in (3.433), $j=0,1,\cdots,s-1$, and the denominator in (3.436). The random vector $\underset{\sim}{\omega}$ was given in (3.429). Under H_a, $\underset{\sim}{\omega}'(Q_{s-1}-Q_{s-2})\underset{\sim}{\omega}$ is distributed like a random variable of the form $\left(\text{see Box (1954a)}\right)$

$$\Upsilon=\sum_{j=1}^{c_{s-1}-c_{s-2}}\pi_j\chi_j, \tag{3.484}$$

where the χ's are independent chi-squared variates with one degree of freedom and the π's are the nonzero eigenvalues of the matrix

$$\mathcal{M}=(Q_{s-1}-Q_{s-2})(\mathrm{Var}(\underset{\sim}{\omega})) \tag{3.485}$$

Similarly to the latter part of the proof to Theorem 3.13(a), we can simplify (3.485) to

$$\mathcal{M}=(Q_{s-1}-Q_{s-2})\left[\sigma^2_{s-1}H_1\tilde{A}_{s-1}H_1'+\delta I_{m-1}\right].$$

The exact distribution of Υ is complicated. To obtain an approximate expression for the power we use the approach described in Hirotsu (1979) (see Appendix D). Let

$$\mathrm{MSE}_{\underset{\sim}{\omega}} = \mathrm{SSE}_{\underset{\sim}{\omega}}/(m - c_{s-1}) \tag{3.486}$$

$$H = \frac{\underset{\sim}{\omega}'(Q_{s-1} - Q_{s-2})\underset{\sim}{\omega}/c^*f}{\mathrm{MSE}_{\underset{\sim}{\omega}}/\delta} \tag{3.478}$$

$$h = \frac{\delta(c_{s-1} - c_{s-2})}{c^*f}\ F_{\alpha, c_{s-1} - c_{s-2}, m - c_{s-1}} \tag{3.488}$$

where in this case

$$c^* = \frac{\mathrm{tr}(\mathcal{M}^2)}{\mathrm{tr}(\mathcal{M})} \tag{3.489}$$

$$f = \frac{\left[\mathrm{tr}(\mathcal{M})\right]^2}{\mathrm{tr}(\mathcal{M}^2)}. \tag{3.490}$$

Then the approximate power is given by

$$\begin{aligned}\Psi_{s-1} = P(H \geq h \mid H_a) &\doteq P(F_{f, f_2} \geq h) + \left[\rho/\{3(f+2)(f+4)B(\tfrac{1}{2}f, \tfrac{1}{2}f_2)\}\right] \\ &\times\left(1 + \frac{fh}{f_2}\right)^{-\frac{1}{2}(f+f_2)}\left(\frac{fh}{f_2}\right)^{\frac{1}{2}f} \\ &\times\left[(f+2)(f+4) - \frac{2(f+f_2)(f+4)}{1+f_2/(fh)} + \frac{(f+f_2+2)(f+f_2)}{\{1+f_2/(fh)\}^2}\right]\end{aligned} \tag{3.491}$$

where

$$f_2 = m - c_{s-1} \tag{3.492}$$

$$\rho = \frac{[\mathrm{tr}(\mathcal{M})][\mathrm{tr}(\mathcal{M}^3)]}{[\mathrm{tr}(\mathcal{M}^2)]^2} - 1 \tag{3.493}$$

and B(a,b) is the beta function.

Similar to the derivation of $E[\underset{\sim}{\omega}'(Q_{s-1}-Q_{s-2})\underset{\sim}{\omega}/(c_{s-1}-c_{s-2})]$ given after (3.441), we can find the trace of $\mathcal{M}$ as follows:

$$\begin{aligned}
\mathrm{tr}(\mathcal{M}) &= \mathrm{tr}[(Q_{s-1}-Q_{s-2})(\sigma^2_{s-1}H_1\tilde{A}_{s-1}H_1' + \delta I_{m-1})] \\
&= \delta(c_{s-1}-c_{s-2}) + \sigma^2_{s-1}\mathrm{tr}[(Q_{s-1}-Q_{s-2})(H_1\tilde{A}_{s-1}H_1')] \\
&= \delta(c_{s-1}-c_{s-2}) + \sigma^2_{s-1}\mathrm{tr}[A_{s-1}'H_1'(Q_{s-1}-Q_{s-2})H_1A_{s-1}] \\
&= \delta(c_{s-1}-c_{s-2}) + \sigma^2_{s-1}\sum_{j=1}^{c_s-1} \underset{\sim}{a}^{*\prime}_{s-1,j}(Q_{s-1}-Q_{s-2})\underset{\sim}{a}^*_{s-1,j}
\end{aligned} \tag{3.492}$$

where, recall

$$\underset{\sim}{a}^*_{s-1,j} = H_1\underset{\sim}{a}_{s-1,j}, \quad j=1,2,\cdots,c-1$$

and

$$A_{s-1} = [\underset{\sim}{a}_{s-1,1}:\cdots:\underset{\sim}{a}_{s-1,c_{s-1}}].$$

In order to evaluate (3.491) we also need $\mathrm{tr}(\mathcal{M}^2)$ and $\mathrm{tr}(\mathcal{M}^3)$. These quantities can be expressed in much the same way as $\mathrm{tr}(\mathcal{M})$ in (3.494). This method of evaluating these traces is similar to the idea of synthesis as described in Milliken and Johnson (1984, p. 219-231).

3.4.8. A Numerical Example

Data were generated according to the following 4-fold random nested model:

$$y_{ijk\ell t} = \mu + \alpha_i + \beta_{ij} + \gamma_{ijk} + \delta_{ijk\ell} + \epsilon_{ijk\ell t},$$

$i = 1,2,3$; $j = 1,2,3$; $k = 1,2,3$; $\ell = 1,2,\cdots,m_{ijk}$; $t = 1,2,\cdots,n_{ijk\ell}$; where μ is an unknown constant parameter; and α_i, β_{ij}, γ_{ijk}, $\delta_{ijk\ell}$, and $\epsilon_{ijk\ell t}$ are independent normally distributed random variables with zero means and variances σ^2_α, σ^2_β, σ^2_γ, σ^2_δ, and σ^2_ϵ, respectively. The m_{ijk} values were randomly generated from the integers $\{2,3,4\}$ and the $n_{ijk\ell}$ values were randomly selected from $\{1,2,3,4\}$. Furthermore, we took $\mu = 10$, $\sigma^2_\alpha = 3$, $\sigma^2_\beta = 0.2$, $\sigma^2_\gamma = 1$, $\sigma^2_\delta = 5$, and $\sigma^2_\epsilon = 1$. These values were only used to generate this (artificial) data set.

Using the methodology developed in Subsections 3.4.3 – 3.4.5, we tested the following hypotheses: $H_0 : \sigma^2_\alpha = 0$, $H_0 : \sigma^2_\beta = 0$, $H_0 : \sigma^2_\gamma = 0$, and $H_0 : \sigma^2_\delta = 0$ versus their usual alternatives. Table 3.16 lists some important quantities used in the development of the exact tests. Table 3.17 summarizes our findings.

3.4.9. Concluding Remarks

In this section we developed exact tests for testing the variance components in a general unbalanced s-fold random nested model where the imbalance affects the last two stages only. In

Table 3.16 Some useful quantities used in the construction of the exact tests for the four variance components in the example

Quantity	Formula cited	Value
s	(3.380)	4
θ_1	(3.381)	$\{i\}$
θ_2	(3.381)	$\{i,j\}$
θ_3	(3.381)	$\{i,j,k\}$
θ_4	(3.381)	$\{i,j,k,\ell\}$
n	(3.385)	216
m	(3.387)	78
c_1	(3.389)	3
c_2	(3.389)	9
c_3	(3.389)	27
$c_4 = m$	(3.389)	78
F_4	(3.407)	9.6053
$\lambda_{\max}$	(3.422)	1.0
δ	(3.428)	$\sigma_\delta^2 + \lambda_{\max}\sigma_\epsilon^2$
F_3	(3.441)	2.5540
$\bar{c}_1$	(3.446)	9
$\bar{c}_2$	(3.446)	3
$\tilde{\underline{B}}_1$	(3.446)	$I_3 \otimes J_9$
$\tilde{\underline{B}}_2$	(3.446)	$I_9 \otimes J_3$
$\lambda_{\max}^*$	(3.464)	0.5
δ^*	(3.465)	$\sigma_\gamma^2 + \lambda_{\max}^*\delta$

Table 3.16 - continued

Quantity	Formula cited	Value
F_2	(3.469)	0.7592
F_1	(3.469)	1.9929

Subsection 3.4.3 we derived an exact test for testing $H_0:\sigma_s^2 = 0$. This test is the usual ANOVA test. Next we considered testing $H_0:\sigma_{s-1}^2 = 0$. In this case, we employed resampling and made a transformation that resulted in a model which resembled an unbalanced (s-1)-fold nested model. Using this fact, we were able to consturct an exact test by deriving a test statistic that was similar to the one used in testing the equivalent null hypothesis in an unbalanced (s-1)-fold nested model in which all the effects are considered fixed.

Finally, in Subsection 3.4.5, we constructed exact tests for testing σ_1^2, $\sigma_2^2, \cdots, \sigma_{s-2}^2$. The motivation behind our procedure came from noting that after making the above transformation, the resulting model resembled an unbalanced random nested model with imbalance affecting the last stage only. Consequently, we could apply the results in Khuri (1990) to aid in the development of our methodology.

From Seely and El-Bassiouni (1983), it is easily seen that the exact test for testing $H_0:\sigma_s^2 = 0$ is a Wald's test associated with model (3.383). Similarly, the exact test for testing $H_0:\sigma_{s-1}^2 = 0$ is a Wald's test associated with model (3.429). Recalling (3.464) and (3.466), let

$$\underset{\sim}{\tau}^{(i)} = (\underset{\sim}{\tau}_1' : \underset{\sim}{\tau}_2' : \cdots : \underset{\sim}{\tau}_i')', \quad i = 1, 2, \cdots, s-1.$$

Table 3.17 Results from our testing procedures

Test for factor	Numerator degrees of freedom	Denominator degrees of freedom	F-value	p-value
A	2	6	F_1=1.9929	0.2169
B(A)	6	18	F_2=0.7592	0.6109
C(AB)	18	51	F_3=2.5540	0.0045
D(ABC)	51	138	F_4=9.6053	≤ 0.0001

Then, the exact test for testing $H_0:\sigma_i^2 = 0$, $i = 1,2,\cdots,s-2$, is a Wald's test based on the model involving only $\underset{\sim}{T}_1, \underset{\sim}{T}_2, \cdots, \underset{\sim}{T}_{i+1}$ (that is, based on $\underset{\sim}{T}^{(i+1)}$).

As a final remark, the extension to more general nested models, including models in which some of the effects are fixed, should offer no resistance. We merely invoke our resampling scheme (when necessary) and make transformations in a "step-down" manner similar to the methodology presented in Section 3.2 and in this section to derive the exact tests.

CHAPTER FOUR
CONCLUSIONS AND DIRECTIONS FOR FUTURE RESEARCH

In the teaching of linear models there seems to be an underlining hierarchy that is followed. The first linear model most students and/or researchers are exposed to is one in which the effects are fixed and the data balanced. For such models, the idea of an “analysis of variance” is used to construct tests of hypotheses concerning the fixed-effects in the model. These tests are easily derived from the corresponding ANOVA table. Furthermore, they enjoy many optimal properties. The next step up this hierarchy brings us to unbalanced fixed-effects models. Now the picture is much more muddled. For example, it is quickly learned that there is no unique partitioning of the total sums of squares (as there is in the balanced data situation) and thus no unique way to construct the ANOVA table. Hence, the sums of squares used in a test of a particular hypothesis must be chosen carefully by the researcher. Otherwise, very spurious results may be obtained.

The concept of a balanced random or mixed model is usually not introduced until after thoroughly analyzing the fixed-effects model. After discussing the differences between these types of models, the focus typically shifts to the problem of constructing point estimators for the variance components in the model. Exact tests of hypotheses concerning the variance components and/or estimable linear combinations of the fixed-effects parameters rest upon finding mean squares that can serve as “error” terms in appropriate F-ratios. However, unlike the balanced fixed-effects case, exact tests (based on the ANOVA mean squares) do not always exist. This phenomenom is even more apparent when the data are unbalanced. Consequently, for unbalanced random or mixed models, tests of hypotheses are usually only developed in the context of an exercise in the application of Satterthwaite's procedure. Therefore, we are left with the idea that in the unbalanced data situation exact tests either do not exist at all or only exist for very simple models. Although Satterthwaite's methodology may provide adequate (approximate) tests when the design is balanced, there is no theoretical justification for its use when the data are unbalanced since in these situations the necessary assumptions will not, in general, be satisfied. The need for exact testing methods is therefore clear. The derivations of some such methods are described in this thesis.

In the development of our procedures, we made the “usual” assumptions about the random-effects. That is we assumed they were independent of each other and had normal distributions with zero means and “$\sigma^2 I$” variance-covariance structure. This led to a particular structure for the Var($\underset{\sim}{y}$). But, are these assumptions correct? Some diagnostic tools would be of help here and perhaps the results of Hocking (1988), Hocking, Green, and Bremer (1989), and Beckman, Nachtsheim, and Cook (1987) could be of use. Also, it would be of interest to develop exact tests under more laxed assumptions. Except for a few special cases, this has not been pursued.

The problem of constructing optimal tests in unbalanced random or mixed models seems to be a difficult one. Besides the results of Spjøtvoll (1967), Mostafa (1967), and Mathew and Sinha (1988), little is known about this area. Even these papers deal only with very simple models. Therefore it is not known if our tests (or tests constructed similarly to ours) possess any optimal properties similar to the ones mentioned in the above articles. A full solution to this problem may be unattainable. Partial solutions may possibly be obtained through intensive computer simulation.

Öfversten (1989) also constructed exact tests for the variance components in unbalanced random and mixed models. His methods differed from ours in the way he initially attacked the problem -- orthogonally transforming the model matrices into layered-triangular form (see Allen (1974)). However, this transformation did not always lead to an exact test. When it did not, Öfversten relied on a resampling scheme similar to ours to aide in the construction of the test. Although his procedure is adaptable to many unbalanced models, it is not clear how to utilize it in models not discussed in his dissertation. Still it would be of interest to compare his results to ours.

A drawback to our procedures, and indeed, to all procedures utilizing a transformation based on a resampling scheme, is that the values of the test statistics are not unique. This is true since only a portion of the error vector is utilized in these transformations. However, the distributional properties of the resulting vector remain unchanged no matter what portion of the error vector is used in the transformation. Consequently, the power of the exact tests will not depend on what portion is used in the transformation. The power will, however, depend on certain design parameters (eigenvalues of certain matrices). Thus, it is of interest to know exactly how much influence these design parameters have on the power. This has yet to be resolved.

In this dissertation we have developed exact procedures to handle the variance components and estimable linear functions of the fixed-effects in some unbalanced random and mixed models. As noted above, there are still many unresolved problems left to tackle. It is our hope that the techniques put forth in this work and the ones sure to be developed elsewhere will help in the construction of solutions to these challenging problems.

APPENDIX A
MATHEMATICAL RESULTS

We present some useful mathematical facts in this appendix. When necessary, proofs will be supplied. Otherwise, the relevant source will be cited. We begin with a result dealing with the rank of a product of matrices

Lemma A.1 (Frobenius Inequality)

Let A, B, and C be matrices of dimensions $n \times p$, $p \times q$ and $q \times t$, respectively. Then

$$\operatorname{rank}(ABC) \geq \operatorname{rank}(AB) + \operatorname{rank}(BC) - \operatorname{rank}(B)$$

Proof:

First note that if either A, B, C, AB, or BC is a zero matrix then the result trivially follows. So, suppose that none of these matrices have rank zero. Then, there exists nonsingular matrices P and Q, both of order $q \times q$, such that

$$P(AB)'(AB)Q = \begin{bmatrix} I_{r_{AB}} & 0 & 0 \\ 0 & 0_{r_B - r_{AB}} & 0 \\ 0 & 0 & 0_{q-r_B} \end{bmatrix} \qquad (A.1)$$

where $r_{AB} = \text{rank}(AB) > 0$ and $0 < r_B = \text{rank}(B) \leq q$.

Define the $q \times q$ matrix X as

$$X = P^{-1} \begin{bmatrix} 0_{r_{AB}} & 0 & 0 \\ 0 & I_{r_B - r_{AB}} & 0 \\ 0 & 0 & 0_{q-r_B} \end{bmatrix} Q^{-1}. \tag{A.2}$$

From (A.1) and (A.2) we have

$$(AB)'(AB) + X = P^{-1} \begin{bmatrix} I_{r_B} & 0 \\ 0 & 0_{q-r_B} \end{bmatrix} Q^{-1}. \tag{A.3}$$

Write Q^{-1} as $Q^{-1} = \begin{bmatrix} Q_1 \\ \cdot\cdot \\ Q_2 \end{bmatrix}$ where Q_1 is of order $r_B \times q$ while Q_2 is of order $(q - r_B) \times q$. Then from (A.3),

$$P^{-1} \begin{bmatrix} I_{r_B} & 0 \\ 0 & 0_{q-r_B} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_1 \end{bmatrix} = P^{-1} \begin{bmatrix} Q_1 \\ 0 \end{bmatrix}. \tag{A.4}$$

Also, write B as

$$B = \begin{bmatrix} B_1 \\ \cdot\cdot \\ B_2 \end{bmatrix} \tag{A.5}$$

where B_1 is of order $r_B \times q$ and B_2 is of order $(p - r_B) \times q$ ($r_B \leq P$ since B is $p \times q$). The r_B rows of B_1 are chosen to be linearly independent. Since Q^{-1} is nonsingular it follows that the r_B rows of Q_1 are also linearly independent. Thus there exists a nonsingular matrix M of dimension $r_B \times r_B$ such that

$$Q_1 = MB_1 \tag{A.6}$$

Now

$$\text{rank}(BC) = \text{rank}(B_1C) = \text{rank}(M^{-1}Q_1C) = \text{rank}(Q_1C).$$

But

$$\text{rank}(Q_1C) = \text{rank}\left(\begin{bmatrix} Q_1 \\ 0 \end{bmatrix} C\right)$$

$$= \text{rank}\left(\begin{bmatrix} I_{r_B} & 0 \\ 0 & 0_{q-r_B} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_1 \end{bmatrix} C\right)$$

$$= \text{rank}\left(P^{-1} \begin{bmatrix} I_{r_B} & 0 \\ 0 & 0_{q-r_B} \end{bmatrix} Q^{-1}C\right)$$

$$= \text{rank}\Big((AB)'ABC + XC\Big) \qquad \Big(\text{see (A.3)}\Big)$$

$$\leq \text{rank}\Big((AB)'ABC\Big) + \text{rank}(XC)$$

$$\leq \text{rank}(ABC) + \text{rank}(X)$$

$$= \text{rank}(ABC) + \text{rank}(B) - \text{rank}(AB) \qquad \big(\text{from (A.2)}\big)$$

Thus,

$$\text{rank}(ABC) \geq \text{rank}(AB) + \text{rank}(BC) - \text{rank}(B)$$

□

Remark A.1.1

Suppose $q = p$ and let B be the $p \times p$ identity matrix. Then, by this lemma we have

$$\text{rank}(AC) \geq \text{rank}(A) + \text{rank}(C) - p \qquad (A.7)$$

Furthermore, if $n = p$ then we get, as a special case to this lemma, Sylvester's law of nullity (see Searle (1982) p. 198).

Remark A.1.2

Most often we will use Lemma A.1 in the following context: A is $(p-1) \times p$ of rank p-1; $B = I_p$; C is $p \times t$ of rank $r \leq p-1$. Further, we will be able to find a vector $\underset{\sim}{x} \neq \underset{\sim}{0}$ such that $C\underset{\sim}{x} \neq \underset{\sim}{0}$ and $AC\underset{\sim}{x} = \underset{\sim}{0}$. Consequently it is clear that

$$\mathrm{rank}(AC) \leq r-1$$

By the Frobenius inequality we will also have

$$\mathrm{rank}(AC) = \mathrm{rank}(A) + \mathrm{rank}(C) - p$$

$$= p-1+r-p = r-1$$

Thus, $\mathrm{rank}(AC) = r-1$.

The next three results deal with the simultaneous diagonalization of two symmetric matrices. The proofs to all three of these lemmas can be found in Searle (1982, pp. 312-314). However, we will present a slightly different proof to Lemma A.4 then what is found in Searle.

Lemma A.2

For symmetric A and B of the same order there exists an orthogonal matrix P such that $P'AP$ and $P'BP$ are both diagonal if and only if $AB = BA$.

Remark A.2.1

Since A and B are both symmetric, the condition $AB = BA$ is equivalent to showing that AB is symmetric.

Lemma A.3

For symmetric matrices A and B of the same order, with A being positive definite (p.d.), there exists a nonsingular matrix M (not necessarily orthogonal), such that $M'AM = I$ and $M'BM$ is a diagonal matrix whose diagonal elements are solutions λ to $|B - \lambda A| = 0$.

Remark A.3.1

The diagonal elements of $M'BM$ are called eigenvalues of B with respect to A. Since A is p.d. it is clear that these diagonal elements are also the eigenvalues of $A^{-1}B$ which are equivalent to the eigenvalues of BA^{-1} which are equivalent to the eigenvalues of $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$.

Lemma A.4

For real symmetric matrices A and B of the same order and non-negative (n.n.d.) there exists a nonsingular matrix P such that $P'AP$ and $P'BP$ are both diagonal.

Proof:

Let the order of A and B be n, with ranks a and b, respectively, and without loss of generality suppose $a \leq b$. Since A is symmetric and n.n.d., there exists a nonsingular matrix T such that

$$T'AT = \begin{bmatrix} I_a & 0 \\ 0 & 0_{n-a} \end{bmatrix}. \tag{A.8}$$

Denote T′BT by

$$T'BT = \begin{bmatrix} E & F \\ F' & G \end{bmatrix} \tag{A.9}$$

where E is a×a and G is (n-a)×(n-a). Now, let C be any (n-a)×a matrix which satisfies

$$F + C'G = 0. \tag{A.10}$$

One such choice for C is

$$C = -G^{-}F' \tag{A.11}$$

where G^{-} is a generalized inverse of G. That is, G^{-} satisfies $GG^{-}G = G$. Consider the nonsingular matrix S defined as

$$S = \begin{bmatrix} I_a & 0 \\ C & M \end{bmatrix} \tag{A.12}$$

where M is an (n-a)×(n-a) nonsingular matrix satisfying

$$M'GM = \begin{bmatrix} I_g & 0 \\ 0 & 0_{n-a-g} \end{bmatrix} \tag{A.13}$$

where $g = \text{rank}(G)$. Now $S'T'ATS = T'AT$ by the construction of S. While, by (A.10) and (A.12), $S'T'BTS$ becomes

$$S'T'BTS = \begin{bmatrix} H & 0 & 0 \\ 0 & I_g & 0 \\ 0 & 0 & 0_{n-a-g} \end{bmatrix} \tag{A.14}$$

where H is the $a \times a$ matrix

$$H = E + C'F'. \tag{A.15}$$

Note that the rank of H is b-g. Since $C'F'$ can also be expressed as $-C'GC$ (see (A.10)), we have that H is symmetric. Thus there exists an orthogonal matrix Q such that

$$Q'HQ = D_a = \text{diag}(\lambda_1, \cdots, \lambda_{b-g}, \ 0, \cdots, 0) \tag{A.16}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_{b-g}$ are the nonzero eigenvalues of H. Therefore, upon defining R as

$$R = \begin{bmatrix} Q & 0 & 0 \\ 0 & I_g & 0 \\ 0 & 0 & I_{n-a-g} \end{bmatrix} \tag{A.17}$$

we have from (A.14) and (A.16),

$$R'S'T'BTSR = \begin{bmatrix} D_a & 0 & 0 \\ 0 & I_g & 0 \\ 0 & 0 & 0_{n-a-g} \end{bmatrix}. \tag{A.18}$$

Also,

$$R'S'T'ASTR = \begin{bmatrix} Q'Q & 0 \\ 0 & 0_{n-a} \end{bmatrix} \tag{A.19}$$

$$= \begin{bmatrix} I_a & 0 \\ 0 & 0_{n-a} \end{bmatrix}. \tag{A.20}$$

Hence, upon letting

$$P = TSR \tag{A.21}$$

we have $P'AP$ and $P'BP$ both diagonal. □

Corollary A.4.1

Suppose the conditions of lemma A.4 hold. Further, if $a = \text{rank}(A)$ and $b = \text{rank}(B)$, suppose that $\text{rank}(A+B) = a+b \leq n$. Then there exists a nonsingular matrix L such that

$$L'AL = \begin{bmatrix} I_a & 0 & 0 \\ 0 & 0_b & 0 \\ 0 & 0 & 0_{n-a-b} \end{bmatrix} \tag{A.22}$$

and

$$L'BL = \begin{bmatrix} 0_a & 0 & 0 \\ 0 & I_b & 0 \\ 0 & 0 & 0_{n-a-b} \end{bmatrix} \tag{A.23}$$

Proof:

From lemma A.4, there exists a nonsingular matrix P such that $P'AP$ and $P'BP$ are both diagonal. Since the $\text{rank}(A+B) = a+b = \text{rank}(A) + \text{rank}(B)$, it follows that the nonzero diagonal elements of $P'AP$ correspond to zero diagonal elements of $P'BP$ and vice-versa. Clearly then, there exists nonsingular matrices M_1 and M_2 for which $M'P'APM$ and $M'P'BMP$ have the forms indicated in (A.22) and (A.23), respectively, where $M = M_1M_2$. Upon defining $L = PM$, the result follows. □

Remark A.4.1

Although not explicitly stated in the proof to lemma A.4 or corollary A.4.1, it should be noted that H, defined in (A.15), is n.n.d. This follows from the fact that $C'F'$ can be written as

$$C'F' = -FG^-F' \tag{A.24}$$

(see (A.11) and use the facts that FG^-F' is invariant to choice of G^- (Corollary 12.2.20.2 of Graybill (1983, p. 416)) and that $E - FG^-F'$ is n.n.d. (Theorem 12.2.21(3) of Graybill (1983, p. 417)). Thus, the eigenvalues of H are nonnegative.

We present one last result concerning the diagonalization of symmetric matrices. This will extend Lemma A.2. The reader is referred to Graybill (1983, p. 408).

Lemma A.5

Let $A_1, A_2, \cdots, A_k$ be symmetric $n \times n$ matrices. A necessary and sufficient condition that there exists an orthogonal matrix P such that $P'A_iP$ is diagonal for all $i = 1, 2, \cdots, k$ is $A_iA_j = A_jA_i$ (or A_iA_j is symmetric) for all i and j.

APPENDIX B
STATISTICAL RESULTS

In this appendix we give some statistical results. Most of the following concepts are well known and are only presented for the sake of completeness. Unless otherwise stated, these can be found in Graybill (1976, pp. 134-140).

Lemma B.1

Let the $n \times 1$ random vector $\underset{\sim}{y}$ be distributed as $N(\underset{\sim}{\mu}, \Sigma)$ where Σ has rank n. If $B\Sigma A = 0$, the quadratic form $\underset{\sim}{y}'A\underset{\sim}{y}$ is independent of the linear form $B\underset{\sim}{y}$ where B is a $q \times n$ matrix.

Lemma B.2

Let the $n \times 1$ random vector $\underset{\sim}{y}$ be distributed as $N(\underset{\sim}{\mu}, \Sigma)$ where Σ has rank n. If $A\Sigma B = 0$, then the two quadratic forms $\underset{\sim}{y}'A\underset{\sim}{y}$ and $\underset{\sim}{y}'B\underset{\sim}{y}$ are independent.

Lemma B.3

Let $\underset{\sim}{y}$ be an $n \times 1$ random vector and let $E(\underset{\sim}{y}) = \underset{\sim}{\mu}$, $\mathrm{Var}(\underset{\sim}{y}) = \Sigma$. Then

$$E(\underset{\sim}{y}'A\underset{\sim}{y}) = tr(A\Sigma) + \underset{\sim}{\mu}'A\underset{\sim}{\mu}$$

The next two results deal with the distribution of a quadratic form.

Lemma B.4

Let the $n\times 1$ random vector $\underset{\sim}{y}$ be distributed as $N(\underset{\sim}{\mu},\Sigma)$ where Σ has rank n. Then the quadratic form $U=\underset{\sim}{y}'A\underset{\sim}{y}\sim\chi^2_p(\lambda)$ where p is the rank(A) and $\lambda=\frac{1}{2}\underset{\sim}{\mu}'A\underset{\sim}{\mu}$, if and only if one of the following three conditions are satisfied: (1) $A\Sigma$ is an idempotent matrix of rank p; (2) ΣA is an idempotent matrix of rank p; (3) Σ is a generalized inverse of A (i.e., $A\Sigma A=A$).

Lemma B.5

Let the $n\times 1$ random vector $\underset{\sim}{y}$ be distributed as $N(\underset{\sim}{\mu},\Sigma)$ where Σ has rank n. The random variable $\underset{\sim}{y}'A\underset{\sim}{y}$ has the same distribution as the random variable U, where $U=\sum_{i=1}^{n} d_{ii}U_i$, where the d_{ii} are the eigenvalues of the matrix $A\Sigma$, and where $U_1,U_2,\cdots,U_n$ are n independent noncentral chi-square random variables, each with one degree of freedom.

Our last lemma applies some of the above results to a general mixed-model. This result is particularly useful since it is used throughout this dissertation.

Lemma B.6

Consider the general mixed-model

$$\underset{\sim}{y} = X\underset{\sim}{g} + Z\underset{\sim}{h} + \underset{\sim}{\epsilon} \tag{B.1}$$

where $X = (X_0 : \cdots : X_{\nu-p})$; $Z = (X_{\nu-p+1} : \cdots : X_\nu)$; $\underset{\sim}{g} = (\underset{\sim}{\beta}'_0 : \cdots : \underset{\sim}{\beta}'_{\nu-p})'$ is the vector of fixed effects; and $\underset{\sim}{h} = (\underset{\sim}{\beta}'_{\nu-p+1} : \cdots : \underset{\sim}{\beta}'_\nu)'$ is the vector of random effects. We will assume that the $n \times 1$ vector of errors, $\underset{\sim}{\epsilon}$, is distributed as $N(\underset{\sim}{0}, \sigma^2_\epsilon I_n)$ independently of $\underset{\sim}{h} \sim N(\underset{\sim}{0}, \Psi)$. Consequently,

$$\underset{\sim}{y} \sim N(X\underset{\sim}{g}, \Sigma) \tag{B.2}$$

where

$$\Sigma = Z\Psi Z' + \sigma^2_\epsilon I_n . \tag{B.3}$$

For ease of notation let $W = [X:Z]$ and let $r = \text{rank}(W)$. Also, define the $n \times n$ matrix R to be

$$R = I_n - W(W'W)^- W' \tag{B.4}$$

Then

(i) R is symmetric, idempotent of rank n-r.

(ii) Clearly $RW = 0$. Since $\text{range}(X_i) \subset \text{range}(W)$ for all $1 = 0,1,\cdots,\nu$, it follows that $RX_i = 0$, $i = 0,1,\cdots,\nu$.

(iii) The error sum of squares can be written as

$$SSE = \underset{\sim}{y}'R\underset{\sim}{y}. \tag{B.5}$$

Since from (i) and (ii), $R\Sigma/\sigma_\epsilon^2 = R$ (which is idempotent) and $RX\underset{\sim}{g} = \underset{\sim}{0}$, we see, using lemma B.4, that

$$SSE/\sigma_\epsilon^2 \sim \chi^2_{n-r}.$$

(iv) From (i), there exists an orthogonal matrix C such that

$$R = C\Lambda C' \tag{B.6}$$

where

$$\Lambda = \text{diag}(I_{\eta_1}, I_{\eta_2}, 0) \tag{B.7}$$

and

$$\eta_1 + \eta_2 = n-r. \tag{B.8}$$

Suppose $\eta_1 > 0$, then for η_2 to be positive we must necessarily assume

$$n > \eta_1 + r. \tag{B.9}$$

Partition C as

$$C = [C_1 : C_2 : C_3] \tag{B.10}$$

where C_1 is $n \times \eta_1$; C_2 is $n \times \eta_2$; and C_3 is $n \times r$. Then

$$\begin{aligned} C_i' C_i &= I \qquad (i = 1,2,3) \\ C_i' C_j &= 0 \qquad i \neq j. \end{aligned} \tag{B.11}$$

Furthermore

$$\begin{aligned} SSE &= \underset{\sim}{y}' R \underset{\sim}{y} = \underset{\sim}{y}' C \Lambda C' \underset{\sim}{y} \\ &= \underset{\sim}{y}' C_1 C_1' \underset{\sim}{y} + \underset{\sim}{y}' C_2 C_2' \underset{\sim}{y} \\ &= SSE_1 + SSE_2 \end{aligned} \tag{B.12}$$

and $SSE_1/\sigma_\epsilon^2 \sim \chi^2_{\eta_1}$ independently of $SSE_2/\sigma_\epsilon^2 \sim \chi^2_{\eta_2}$.

This forms the basis behind the idea used in resampling from the error vector.

(v) Suppose $\text{range}(X_i) \subset \text{range}(X_\nu)$ for all $i = 0, 1, \cdots, \nu\text{-}1$. Define the matrix D as

$$D = (X_\nu' X_\nu)^{-1} X_\nu'. \tag{B.13}$$

Further, let the random vector $\underset{\sim}{x}$ be

$$\underset{\sim}{x} = D\underset{\sim}{y}. \tag{B.14}$$

Then, by lemma B.1, since

$$D\Sigma R = \sigma_\epsilon^2 DR = 0 \tag{B.15}$$

we have that $\underset{\sim}{x} = D\underset{\sim}{y}$ and $SSE = \underset{\sim}{y}'R\underset{\sim}{y}$ are independent. (The fact that $DR = 0$ follows after noting that range(X_ν) $\subset$ range(W) and $RW = 0$.)

APPENDIX C
AN ILLUSTRATION OF THE UTILITY OF RESAMPLING

In this appendix we consider the general unbalanced mixed cross-classification model to illustrate the idea of resampling from the error vector. We assume the imbalance affects the last stage only. Thus, this model is a special case of model (3.189) in Section three of Chapter three.

As it applies to models having imbalance in the last stage only, resampling serves to reduce the analysis of the unbalanced model to that of a balanced one. For designs having more general forms of imbalance, resampling will typically need to be employed more than once to construct exact tests. This is the case for the designs studied in Sections two and four of Chapter three.

We start by considering the following model:

$$\underset{\sim}{y} = X\underset{\sim}{h} + Z\underset{\sim}{g} + \underset{\sim}{\epsilon} \tag{C.1}$$

where $X = (X_0 : \cdots : X_{\nu-p})$; $Z = (X_{\nu-p+1} : \cdots : X_\nu)$; $\underset{\sim}{h} = (\underset{\sim}{\beta}_0' : \cdots : \beta_{\nu-p}')'$ is the vector of unknown parameters; $\underset{\sim}{g} = (\beta_{\nu-p+1}' : \cdots : \underset{\sim}{\beta}_\nu')'$ is the vector of random effects, $\underset{\sim}{y}$ is the $n \times 1$ vector of observations; and $\underset{\sim}{\epsilon}$ is the $n \times 1$ vector of errors. We assume that $\underset{\sim}{\beta}_{\nu-p+1}, \cdots, \underset{\sim}{\beta}_\nu$ and $\underset{\sim}{\epsilon}$ are independent, normally distributed random vectors with zero mean vectors and variance-covariance matrices given by

$$\begin{aligned} \operatorname{Var}(\underset{\sim}{\beta}_i) &= \sigma_i^2 I_{c_i} \qquad (i = \nu\text{-}p+1, \cdots, \nu) \\ \operatorname{Var}(\underset{\sim}{\epsilon}) &= \sigma_\epsilon^2 I_n \end{aligned} \tag{C.2}$$

where c_i is the number of elements in $\underset{\sim}{\beta}_i$ $(i = 0, 1, \cdots, \nu)$.

The vector of cell means, $\underset{\sim}{\bar{y}}$, can be expressed as

$$\underset{\sim}{\bar{y}} = D\underset{\sim}{y} \tag{C.3}$$

where

$$D = (X_\nu' X_\nu)^{-1} X_\nu'. \tag{C.4}$$

we can also express (C.3) as

$$\underset{\sim}{\bar{y}} = H\underset{\sim}{h} + G\underset{\sim}{g} + \underset{\sim}{\bar{\epsilon}} \tag{C.5}$$

where $H = (A_0 : \cdots : A_{\nu-p})$ and $G = (A_{\nu-p+1} : \cdots : A_\nu = I_{c_\nu})$. Also, $\underset{\sim}{\bar{\epsilon}} = D\underset{\sim}{\epsilon}$. The A_i $(i = 0, 1, \cdots, \nu\text{-}1)$ are described in Khuri (1990). In this same article, Khuri mentions that the matrices $B_i = A_i A_i'$ $(i = 0, 1, \cdots, \nu)$ commute in pairs. Consequently, by lemma A.5, there exists an orthogonal matrix Q which simultaneously diagonalizes them. Write Q as

$$Q = \begin{bmatrix} Q_1 \\ \cdot\ \cdot \\ Q_2 \end{bmatrix} \tag{C.6}$$

where Q_2 is constructed so that

$$Q_2 H = 0. \tag{C.7}$$

From (C.5) and the distributional properties of $\underset{\sim}{g}$ and $\underset{\sim}{\epsilon}$, it can be easily shown that

$$\begin{aligned} E(\underset{\sim}{\bar{y}}) &= H\underset{\sim}{h} \\ \operatorname{Var}(\underset{\sim}{\bar{y}}) &= \sum_{i=\nu-p+1}^{\nu} \sigma_i^2 A_i A_i' + \sigma_\epsilon^2 (X_\nu' X_\nu)^{-1} \\ &= \sum_{i=\nu-p+1}^{\nu-1} \sigma_i^2 B_i + \sigma_\nu^2 I_{c_\nu} + K\sigma_\epsilon^2 \end{aligned} \tag{C.8}$$

where

$$K = (X_\nu' X_\nu)^{-1}. \tag{C.9}$$

Now, define the $q \times 1$ random vector $\underset{\sim}{u}$ as

$$\underset{\sim}{u} = Q_2 \underset{\sim}{\bar{y}} \tag{C.10}$$

where q is the number of rows of Q_2. Then

$$\begin{aligned} E(\underset{\sim}{u}) &= Q_2 H \underset{\sim}{h} = \underset{\sim}{0} \\ \operatorname{Var}(\underset{\sim}{u}) &= \sum_{i=\nu-p+1}^{\nu-1} \sigma_i^2 Q_2 B_i Q_2' + \sigma_\nu^2 I_q + Q_2 K Q_2' \sigma_\epsilon^2. \end{aligned} \tag{C.11}$$

If not for the $Q_2KQ_2'\sigma_\epsilon^2$ term in the expression for Var($\underline{u}$) in (C.11), we could analyze model (C.10) as a balanced model. This is where the utility of resampling appears. Coupled with the results of Lemma B.6, resampling will allow us to reduce the analysis of model (C.10) to that of a balanced model.

From (C.1), write

$$W = [X:Z] \tag{C.12}$$

and

$$R = I_n - W(W'W)^- W' \tag{C.13}$$

Assuming

$$n > q + c_\nu \tag{C.14}$$

and using lemma B.6(iv), we can write R as

$$R = C\Lambda C' \tag{C.15}$$

where C is orthogonal and Λ is diagonal. Furthermore, we can partition C and Λ as

$$C = [C_1 : C_2 : C_3]$$

$$\Lambda = \mathrm{diag}(I_q, I_{n-c_\nu-q}, 0) \tag{C.16}$$

where C_1 is $n \times q$; C_2 is $n \times (n-c_\nu-q)$; and C_3 is $n \times c_\nu$. Define the $q \times 1$ random vector $\underset{\sim}{\omega}$ as

$$\underset{\sim}{\omega} = \underset{\sim}{u} + (\lambda_{\max} I_q - L)^{\frac{1}{2}} C_1' \underset{\sim}{y} \tag{C.17}$$

where $L = Q_2 K Q_2'$ and $\lambda_{\max}$ is its largest eigenvalue. C_1 is defined in (C.16). Notice that $\lambda_{\max} I_q - L$ is positive semidefinite and hence $(\lambda_{\max} I_q - L)^{\frac{1}{2}}$ is well-defined. From lemma B.6 the following facts are apparent:

(i) $\underset{\sim}{u}$ is independent of $C_1' \underset{\sim}{y}$.

(ii) $C_1' \Sigma C_1 = \sigma_\epsilon^2 I_q$.

(C.18)

Consequently, we obtain from (C.11):

$$E(\underset{\sim}{\omega}) = \underset{\sim}{0}$$

$$\mathrm{Var}(\underset{\sim}{\omega}) = \sum_{i=\nu-p+1}^{\nu-1} \sigma_i^2 Q_2 B_i Q_2' + \delta I_q \tag{C.19}$$

where $\delta = \sigma_\nu^2 + \sigma_\epsilon^2 \lambda_{\max}$. Thus, resampling has reduced the analysis of the unbalanced model (C.10) to that of a balanced one (C.17). $\Big($See, for example, Searle (1971), Graybill (1976), or Khuri (1982) for a discussion on the analysis of balanced models.$\Big)$

In (C.17) note that $C_1'\underset{\sim}{y}$ is a portion of the error vector, $R\underset{\sim}{y}$. The actual value of $C_1'\underset{\sim}{y}$ depends on which q columns are chosen from the first $n - c_\nu$ columns of C to form C_1 $\Big($see (C.16)$\Big)$. However, the distribution of $\underset{\sim}{\omega}$ will not depend on the choice of C_1. Thus, this approach is similar in spirit to methodologies employing resampling, such as the jackknife and bootstrap.

APPENDIX D
HIROTSU'S APPROXIMATION

In this Appendix we describe the F approximation (used in some of our power calculations) as developed in Hirotsu (1979). We use his notation.

Let

$$\underset{\sim}{x} = N(\underset{\sim}{\mu}, \Sigma)$$

and let $\chi^2_{f_2}$ be a χ^2 variable distributed independently of $\underset{\sim}{x}$. Further, take $\underset{\sim}{\mu} = \underset{\sim}{0}$ (as in the case in our approximations). Then we are concerned with finding $P(Z \geq z)$ where

$$Z = \frac{(\underset{\sim}{x}'A\underset{\sim}{x})/cf}{\chi^2_{f_2}/f_2}, \tag{D.1}$$

and A is a nonnegative definite matrix. The values of c and f are given by

$$\begin{aligned} c &= \tfrac{1}{2}\kappa_2(\underset{\sim}{x}'A\underset{\sim}{x})/\kappa_1(\underset{\sim}{x}'A\underset{\sim}{x}) \\ f &= 2\kappa_1^2(\underset{\sim}{x}'A\underset{\sim}{x})/\kappa_2(\underset{\sim}{x}'A\underset{\sim}{x}) \end{aligned} \tag{D.2}$$

where $\kappa_i(\underset{\sim}{x}'A\underset{\sim}{x})$ represents the i^{th} cumulant of $\underset{\sim}{x}'A\underset{\sim}{x}$. In this situation, these quantities become:

$$c = \mathrm{tr}[(A\Sigma)^2]/\mathrm{tr}(A\Sigma)$$

$$f = [\mathrm{tr}(A\Sigma)]^2/\mathrm{tr}[(A\Sigma)^2]. \tag{D.3}$$

Hirotsu's approximation is then given by

$$\begin{aligned} P(Z \geq z) \doteq{} & P(F_{f_1,f_2} \geq z) \\ & + [\delta/\{3(f+2)(f+4)B(\tfrac{1}{2}f,\tfrac{1}{2}f_2)\}] \\ & \times (1+fz/f_2)^{-\frac{1}{2}(f+f_2)}(fz/f_2)^{\frac{1}{2}f} \\ & \times\left[(f+2)(f+4) - \frac{2(f+f_2)(f+4)}{1+f_2/(fz)} + \frac{(f+f_2+2)(f+f_2)}{\{1+f_2/(fz)\}^2}\right] \end{aligned} \tag{D.4}$$

where B(·,·) is the beta function and

$$\begin{aligned} \delta &= \frac{\frac{1}{2}\kappa_1(\underset{\sim}{x}'A\underset{\sim}{x})\kappa_3(\underset{\sim}{x}'A\underset{\sim}{x})}{\kappa_2^2(\underset{\sim}{x}'A\underset{\sim}{x})} - 1 \\ &= \frac{\{\mathrm{tr}(A\Sigma)\}\{\mathrm{tr}[(A\Sigma)^3]\}}{\{\mathrm{tr}[(A\Sigma)^2]\}^2} - 1. \end{aligned} \tag{D.5}$$

The approximation is based on a generalized Laguerre polynomial expansion of the true distribution of $(\underset{\sim}{x}'A\underset{\sim}{x})/2c$ and has been shown to provide satisfactory results.

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BIOGRAPHICAL SKETCH

I was born in Ft. Wayne, Indiana, on November 4, 1960. When I was six years of age we moved to California. I resided there for 20 years. In 1986 I came to Gainesville to continue my education at the University of Florida. I am engaged to be married to Debra Thurman in August of 1991.

My B.S. degree was in applied mathematics with an emphasis in astronomy and was received from San Diego State University in December of 1983. My M.S. degree, in statistics, was also awarded from San Diego State. I received this degree in August of 1986.

From August of 1986 until August of 1988 I worked in the Biostatistical Unit located in Shands hospital. While there I worked under Dr. Randy Carter and Dr. Jane Pendergast as a statistical consultant. From August 1988 to the present I have been employed as a teaching assistant in the Department of Statistics and have taught many of the undergraduate courses.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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This dissertation was submitted to the Graduate Faculty of the Department of Statistics in the College of Liberal Arts and Sciences and to the Graduate School and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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